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AD830583

REPORT NO. CR-56

DATE March, 1959

NO. OF PAGES 40

CONVAIR | ASTRONAUTICS

CONVAIR DIVISION OF GENERAL DYNAMICS CORPORATION

ORBITAL AND DYNAMIC ELEMENTS FOR SIMPLIFIED TWO BODY PROBLEMS

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I. SUMMARY

The solution to the simplified two body problem is presented based on the parameters of the radial and normal components of velocity made dimensionless by division by the circular velocity at injection . The orbital and dynamic elements are presented in graphical form for velocities up to ten times the circular velocity at injection.

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IV. NOMENCLATURE

A	Dimensionless semi-major/semi-transverse axis
B	Dimensionless semi-minor/semi-conjugate axis
C	Linear eccentricity, constant
D	Directrix distance
E	Eccentricity
f	Velocity parameter (See equation 15)
F	Focus, force
g	Gravitational constant
i	Infinitestimal
m	Mass of revolving body
M	Mass of attracting body
p	Dimensionless semilatus rectum
r	Distance between bodies
R	Dimensionless distance
S	System
t	Time
v	Velocity
V	Dimensionless velocity

α Dimensionless radial velocity

β Dimensionless normal velocity

γ Velocity vector angle with respect to the normal direction

θ Angular polar coordinate

ϕ True anomaly

Subscripts

A Apogee

c Circular

o Initial condition

P Perigee

r Radial component

θ Normal component

Miscellaneous

$\dot{x}$ First derivative of x with respect to time

$\ddot{x}$ Second derivative of x with respect to time

$\bar{x}$ Vector x

$\hat{x}$ Unit vector in x direction

$\equiv$ Equality by definition

$::$ Consists of

V. INTRODUCTION

A. ASSUMPTIONS

The system defining the simplified two body problem consists of two bodies: one, the attracting body considered as a point mass, M; the other the revolving body considered as a point with infinitesimal mass, m . An attraction force is exerted on each body according to Newton's Inverse Square Law of Gravity. The assumptions mathematically represented are:

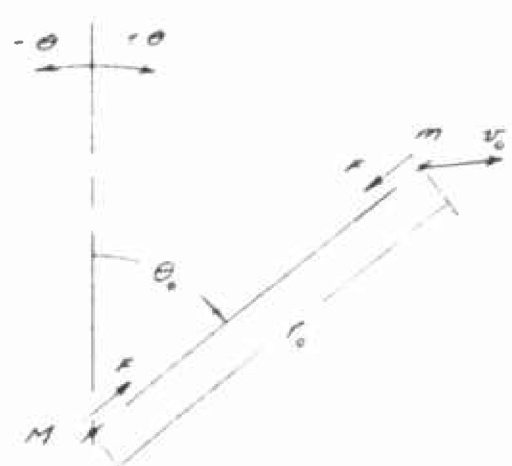
$$S :: M$$

such that: $m = i$

$$\bar{F} = -\frac{GMm}{r^2}\hat{r} \qquad 1$$

B. NECESSARY INITIAL CONDITIONS

To obtain a complete solution to a particular two body problem, the following must be specified: the mass of the attracting body, M ; the distance between the two bodies, r_0 ; the angular distance from some reference line, θ_0 ; the time, t_0 ; the velocity of the revolving body with respect to the attracting body, v_0 ; and an angle, taken as the angle the velocity vector makes with a normal to the connecting line between the two bodies, γ_0 .



To simplify the solution, the following conventions are made:

1. Time is taken as zero for the initial point.
2. All distances are made dimensionless by division by the initial distance.
3. All velocities are made dimensionless by division by the velocity at the initial distance.

C. SELECTION AND DEFINITION OF PARAMETERS

The criteria used in selecting the parameters is the potential to communicate all the orbital information in the most explicit manner. To universalize the solution, the additional stipulation is made that there cannot be any direct dependency on the magnitude of the attracting body.

The satisfying of the latter requirement is best accomplished by utilizing the circular velocity corresponding to the initial distance, which involves both the magnitude of the attracting body and the initial distance.

For a particular M and r_o, any v_o, γ_o combination uniquely determines the orbit. It is thus seen that two parameters are sufficient if one of them involves the above mentioned circular velocity. The parameter combination V_o, γ_o, where

$$V_o = \frac{v_o}{v_{c_o}} = \frac{v_o}{\left(\frac{GM}{r_o}\right)^{\frac{1}{2}}} \qquad 2$$

is the obvious choice, but has the disadvantage of different units.

To form a consistently defined combination which is indicative of the initial injection conditions, the above dimensionless velocity is resolved in a radial direction (α) and a normal direction (β).

$$\begin{aligned} \alpha &\equiv V_o \sin \gamma_o \\ \beta &\equiv V_o \cos \gamma_o \end{aligned} \qquad 3$$

The initial launch angle and velocity become

$$\gamma_0 = \tan^{-1} \frac{\alpha}{\beta} \qquad 4$$

$$V_0 = (\alpha^2 + \beta^2)^{\frac{1}{2}}$$

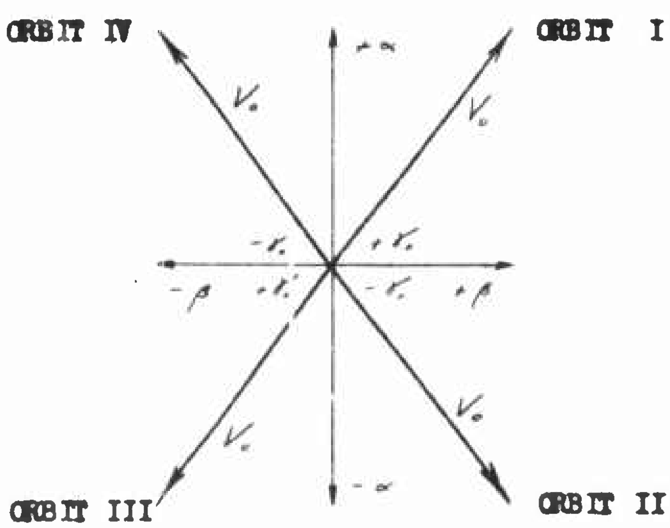
There are four unique combinations of α, β, which define four orbits characterized by:

Orbit I: $+\alpha, \; +\beta, \; +\gamma_0$

Orbit II: $-\alpha, \; +\beta, \; -\gamma_0$

Orbit III: $-\alpha, \; -\beta, \; +\gamma_0$

Orbit IV: $+\alpha, \; -\beta, \; -\gamma_0$



D. EQUATIONS OF MOTION

The forces acting on the body can be resolved into components consisting of a force in the radial direction and a force in the normal direction. The forces acting on the revolving body can be expressed:

$$F_r = -\frac{GMm}{r^2}$$
$$F_\theta = 0 \qquad 5$$

where the total force is represented by

$$\bar{F} = F_r \hat{r} + F_\theta \hat{\theta} \qquad 6$$

According to Newton's Second Law, the force is equal to the mass times the acceleration. Using the familiar expression for acceleration in polar coordinates, the force is

$$\bar{F} = m(\ddot{r} - r\dot{\theta}^2)\hat{r} + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta} \qquad 7$$

VI. BASIC ANALYSIS

A. NORMAL VELOCITY

Comparing 5) and 6), the normal force term in 7) can be set equal to zero

$$0 = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \qquad 8$$

The quantity in the parenthesis can also be written

$$\frac{1}{r}\frac{d(r^2\dot{\theta})}{dt} = 0 \qquad 9$$

As the differential of a constant is zero, 9) reduces to

$$C = r^2\dot{\theta} \qquad 10$$

Applying the general definition of the normal velocity given in 3) for the initial conditions

$$r_o v_\theta = v_{c_o}\beta r_o = C \qquad 11$$

In general, the normal dimensionless velocity is

$$V_\theta = \frac{r\dot{\theta}}{v_{c_o}} = \beta R^{-1} \qquad 12$$

where

$$R = \frac{r}{r_o}$$

B. RADIAL VELOCITY

Setting the radial force term in 7) equal to that found in 5)

$$-\frac{GMm}{r^2} = m(\ddot{r} - r\dot{\theta}^2) \qquad 13$$

Eliminating $\dot{\theta}$ by using 10), and introducing the circular velocity from 2), 13) can be solved for $\dot{r}$

$$\ddot{r} = \frac{r_0^2 v_{c_0}^2 \beta^2}{r^3} - \frac{r_0 v_{c_0}^2}{r^2} \qquad 14$$

Integrating, eliminating the constant by letting $r = r_0$ when $\dot{r} = \alpha v_{c_0}$, the final equation for the radial velocity is

$$V_r = \frac{\dot{r}}{v_{c_0}} = \pm\left(-R^{-2}\beta^2 + 2R^{-1} - f\right)^{\frac{1}{2}} \qquad 15$$

where, the velocity parameter f is

$$f \equiv 2 - \alpha^2 - \beta^2$$

For the initial radial velocity, the sign is chosen to match the sign of α. For other than the initial condition, a check must be made to determine whether the body is on the side of the orbit where the radius is increasing in the direction of motion, or vice versa.

C. PERIGEE ANGLE

In order to obtain r as a function of θ, $\frac{dr}{d\theta}$ is first obtained from

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} \qquad 16$$

Using 12) and 15), and noting that $\frac{dr}{r^2} = -d\left(\frac{1}{r}\right)$:

$$d\theta = -\frac{d(R^{-1}\beta)}{\pm\left(-R^{-2}\beta^2 + 2R^{-1} - f\right)^{\frac{1}{2}}} \qquad 17$$

The standard integral formula

$$\int \frac{dx}{(a^2 - x^2)^{\frac{1}{2}}} = -\cos^{-1}\frac{x}{a} \qquad 18$$

applies when 17) is put in the following form

$$d\theta = \pm\frac{-d(R^{-1}\beta - \beta^{-1})}{\left[\beta^{-2} - f - (R^{-1}\beta - \beta^{-1})^2\right]^{\frac{1}{2}}} \qquad 19$$

The perigee distance is given below and its use here before it is derived is allowed as the result of this section does not affect the analysis leading to its derivation.

$$R_p = \frac{\beta^2}{1 + (1 - f\beta^2)^{1/2}} \qquad 20$$

Integrating 19) between the limits of the initial point and perigee results in

$$\theta_p - \theta_0 = \pm \left[\cos^{-1} \frac{(1 - f\beta^2)^{1/2}}{\beta(\beta^{-2} - f)^{1/2}} - \cos^{-1} \frac{\beta^2 - 1}{\beta(\beta^{-2} - f)^{1/2}} \right] \qquad 21$$

Investigating the result for the four possible orbits, using

$$\cos^{-1}(-x) = \pi - \cos^{-1} x \qquad 22$$

it is found that Orbits I and III and Orbits II and IV have the same perigee angles. By introducing the arbitrary convention of measuring the perigee angle in the opposite direction from which the body is traveling, the perigee angles become

Orbit I : $\theta_p = -\cos^{-1} \dfrac{\beta^2 - 1}{(1 - f\beta^2)^{1/2}} + \theta_0$

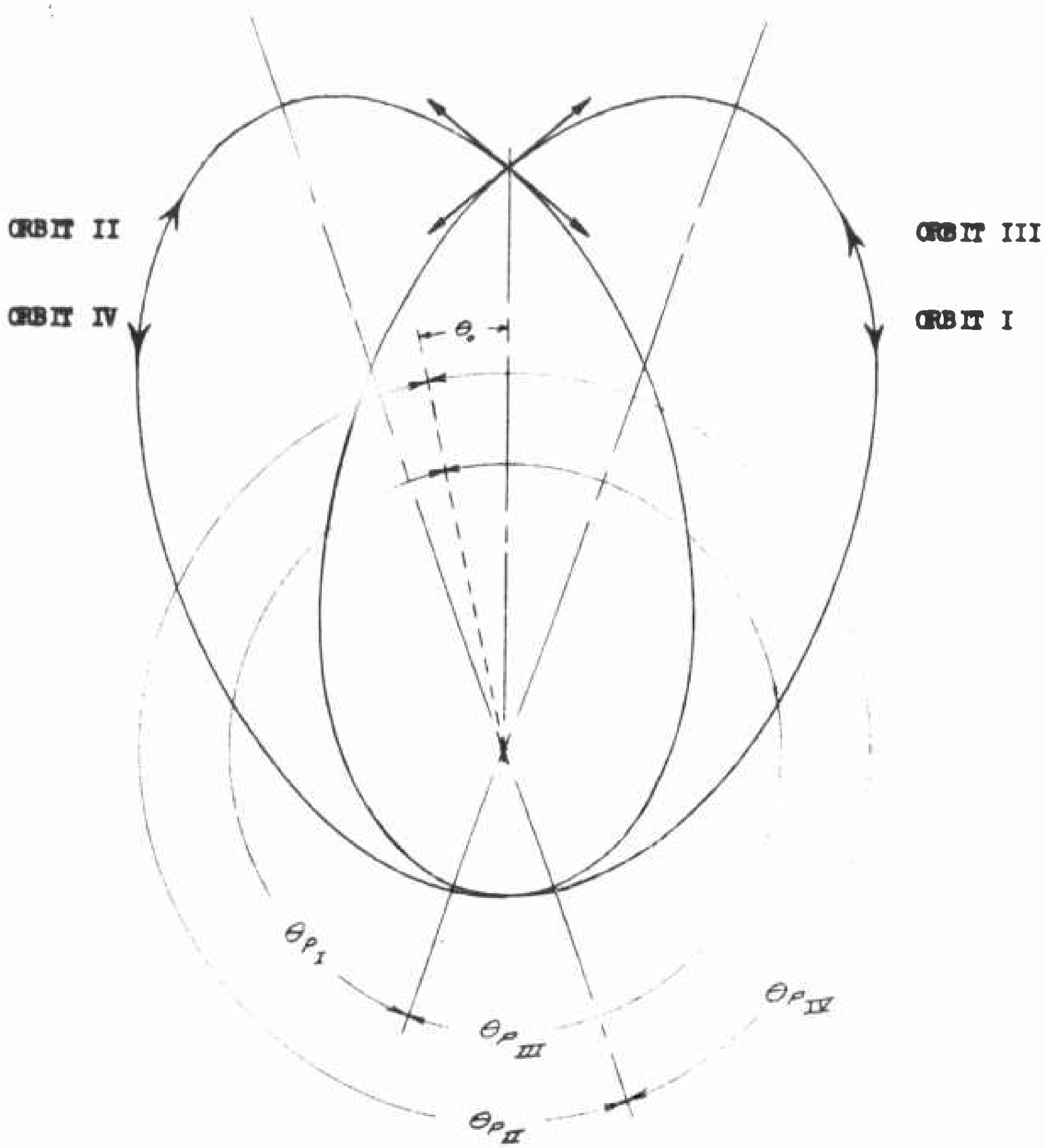
Orbit II : $\theta_p = -2\pi + \cos^{-1} \dfrac{\beta^2 - 1}{(1 - f\beta^2)^{1/2}} + \theta_0$

23

Orbit III : $\theta_p = 2\pi - \cos^{-1} \dfrac{\beta^2 - 1}{(1 - f\beta^2)^{1/2}} + \theta_0$

Orbit IV : $\theta_p = \cos^{-1} \dfrac{\beta^2 - 1}{(1 - f\beta^2)^{1/2}} + \theta_0$

For Orbits I and II, θ is always measured in the plus direction; for Orbits III and IV, θ is measured in the minus direction.



D. RADIUS AS A FUNCTION OF TRUE ANOMALY

Integrating 19) between the limits of the initial point and a general point

$$\theta - \theta_o = \pm \left[\cos^{-1} \frac{R^{-1}\beta^2 - 1}{\beta(\beta^{-2} - f)^{1/2}} - \cos^{-1} \frac{\beta^2 - 1}{\beta(\beta^{-2} - f)^{1/2}} \right] \qquad 24$$

Considering the four possible orbits and the corresponding initial perigee angles, 24) can be reduced to

$$\phi = \cos^{-1} \frac{R^{-1}\beta^2 - 1}{(1 - f\beta^2)^{1/2}}$$

or 25

$$R = \frac{\beta^2}{1 + (1 - f\beta^2)^{1/2} \cos \phi}$$

for all four orbits. The true anomaly ϕ is given for the four Orbits by

Orbit I $\quad \phi = \theta - \theta_p$

Orbit II $\quad \phi = \theta - \theta_p$

Orbit III $\quad \phi = -\theta + \theta_p$ 26

Orbit IV $\quad \phi = -\theta + \theta_p$

The true anomaly determines whether or not the body is on the side of the orbit where the radius is increasing in the direction of motion, or vice versa. For positive $\dot{r}$

$$2\pi n < \phi < \pi + 2\pi n \qquad 27$$

For negative $\dot{r}$

$$\pi + 2\pi n < \phi < 2\pi + 2\pi n \qquad 28$$

where n is the number of times the body has passed perigee.

E. DYNAMIC AND ORBITAL ELEMENTS

Equation 25) can be compared with the standard equation of the conic section in polar coordinates in order to determine the elements of the conic section. The results are shown in Table I.

The calculation of the dynamic elements is straight forward and summarized in Table 2.

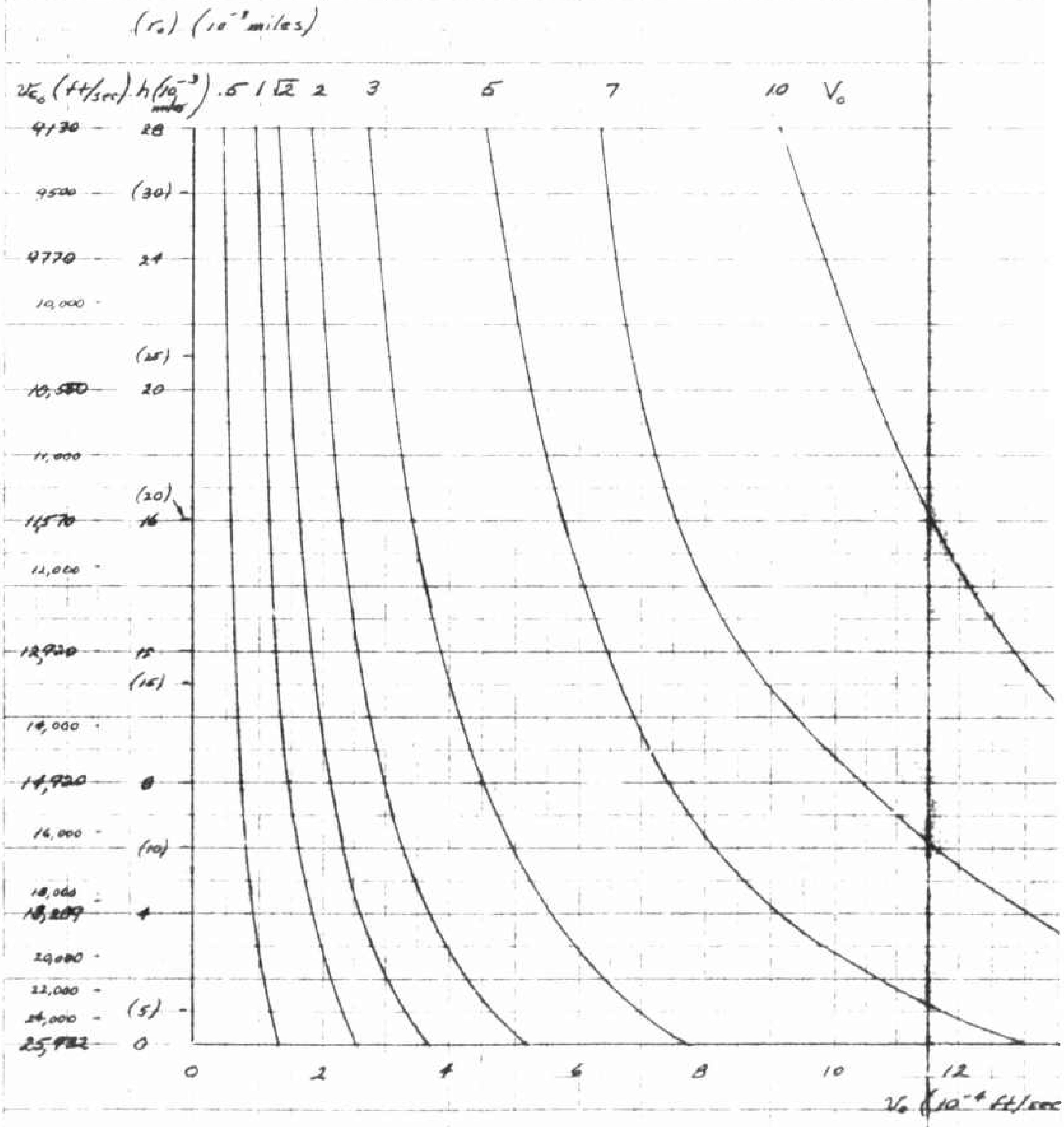
It should be noted that each α, β figure is also a plot of V_0, γ_0, as can be seen from Figure 6.

e	$\left[\alpha^2\beta^2+(\beta^2-1)^2\right]^{\frac{1}{2}}$
p	β^2
A	$\dfrac{1}{\lvert 2-\alpha^2-\beta^2\rvert}$
B	$\dfrac{\lvert\beta\rvert}{\left[\lvert 2-\alpha^2-\beta^2\rvert\right]^{\frac{1}{2}}}$
C	$\dfrac{\left[\alpha^2\beta^2+(\beta^2-1)^2\right]^{\frac{1}{2}}}{\lvert 2-\alpha^2-\beta^2\rvert}$
D	$\dfrac{\beta^2}{\left[\alpha^2\beta^2+(\beta^2-1)^2\right]^{\frac{1}{2}}}$
$\dfrac{B}{A}$	$\lvert\beta\rvert\left[\alpha^2+\beta^2-2\right]^{\frac{1}{2}}$
R_P	$\dfrac{\beta^2}{1+\left[\alpha^2\beta^2+(\beta^2-1)^2\right]^{\frac{1}{2}}}$
R_A	$\dfrac{\beta^2}{1-\left[\alpha^2\beta^2+(\beta^2-1)^2\right]^{\frac{1}{2}}}$

VII. TABLE 1. ORBITAL ELEMENTS

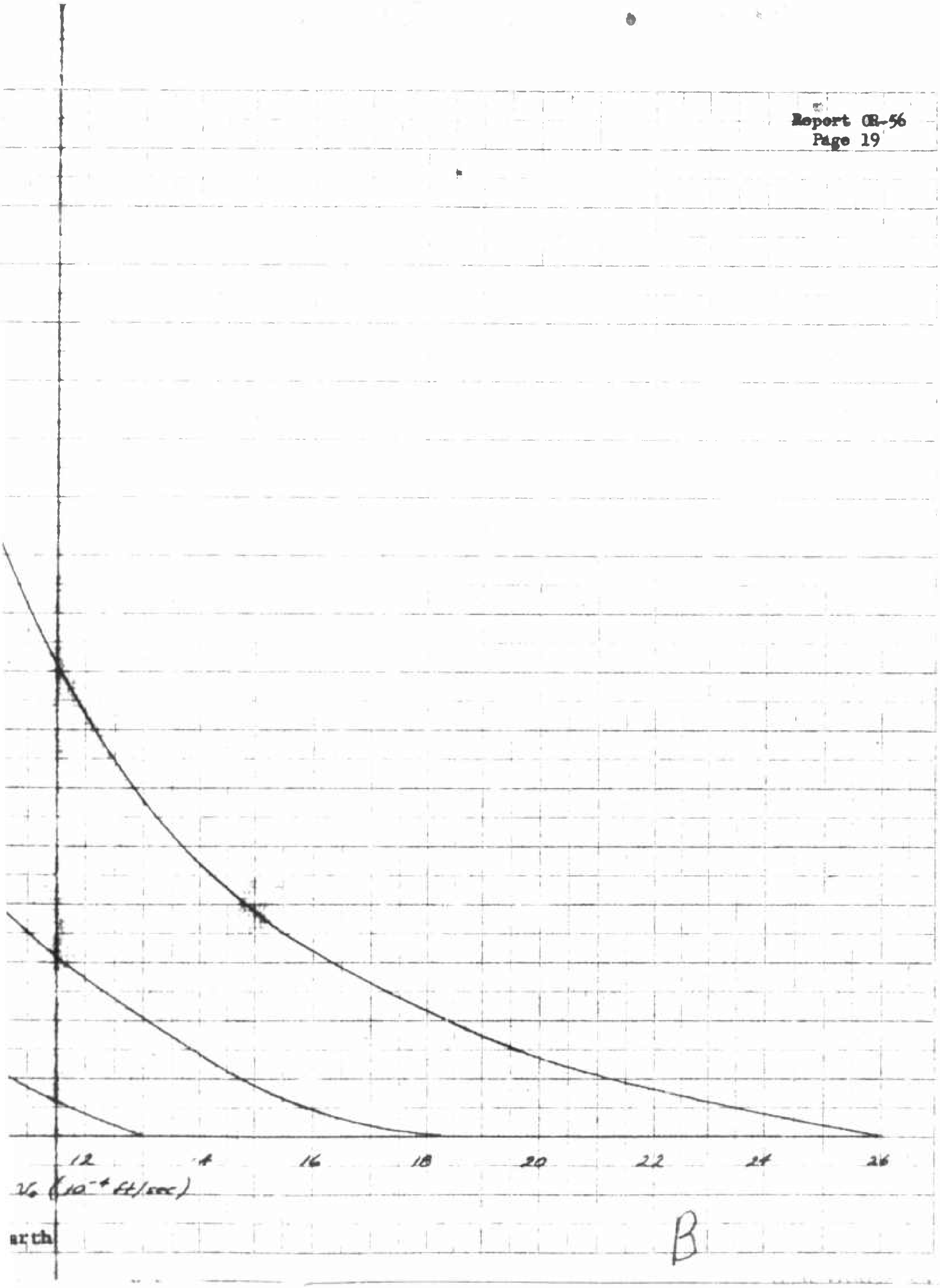
V $f(R^{-1})$	$\left[2R^{-1} - 2 + \alpha^2 + \beta^2\right]^{\frac{1}{2}}$
V $f(\phi)$	$\left\{\dfrac{2\left(1 + \left[\alpha^2\beta^2 + (\beta^2-1)^2\right]^{\frac{1}{2}} \cos\phi\right)}{\beta^2} - 2 + \alpha^2 + \beta^2\right\}^{\frac{1}{2}}$
V_r $f(R^{-1})$	$\left[-R^{-2}\beta^2 + 2R^{-1} - 2 + \alpha^2 + \beta^2\right]^{\frac{1}{2}}$
V_r $f(\phi)$	$\dfrac{\left[\alpha^2\beta^2 + (\beta^2-1)^2\right]^{\frac{1}{2}} \sin\phi}{\beta}$
V_θ $f(R^{-1})$	$R^{-1}\beta$
V_θ $f(\phi)$	$\dfrac{1 + \left[\alpha^2\beta^2 + (\beta^2-1)^2\right]^{\frac{1}{2}} \cos\phi}{\beta}$
V_{θ_P}	$\dfrac{1 + \left[\alpha^2\beta^2 + (\beta^2-1)^2\right]^{\frac{1}{2}}}{\beta}$
V_{θ_A}	$\dfrac{1 - \left[\alpha^2\beta^2 + (\beta^2-1)^2\right]^{\frac{1}{2}}}{\beta}$
γ $f(R^{-1})$	$\tan^{-1} \dfrac{\left[-R^{-2}\beta^2 + 2R^{-1} - 2 + \alpha^2 + \beta^2\right]^{\frac{1}{2}}}{R^{-1}\beta}$
γ $f(\phi)$	$\tan^{-1} \dfrac{\left[\alpha^2\beta^2 + (\beta^2-1)^2\right]^{\frac{1}{2}} \sin\phi}{1 + \left[\alpha^2\beta^2 + (\beta^2-1)^2\right] \cos\phi}$

VIII. TABLE 2. DYNAMIC ELEMENTS



A

Figure 1. Dimensionless Velocity for Earth


12
14
16
18
20
22
24
26
v_0 (10^{-4} ft/sec)
arth
B

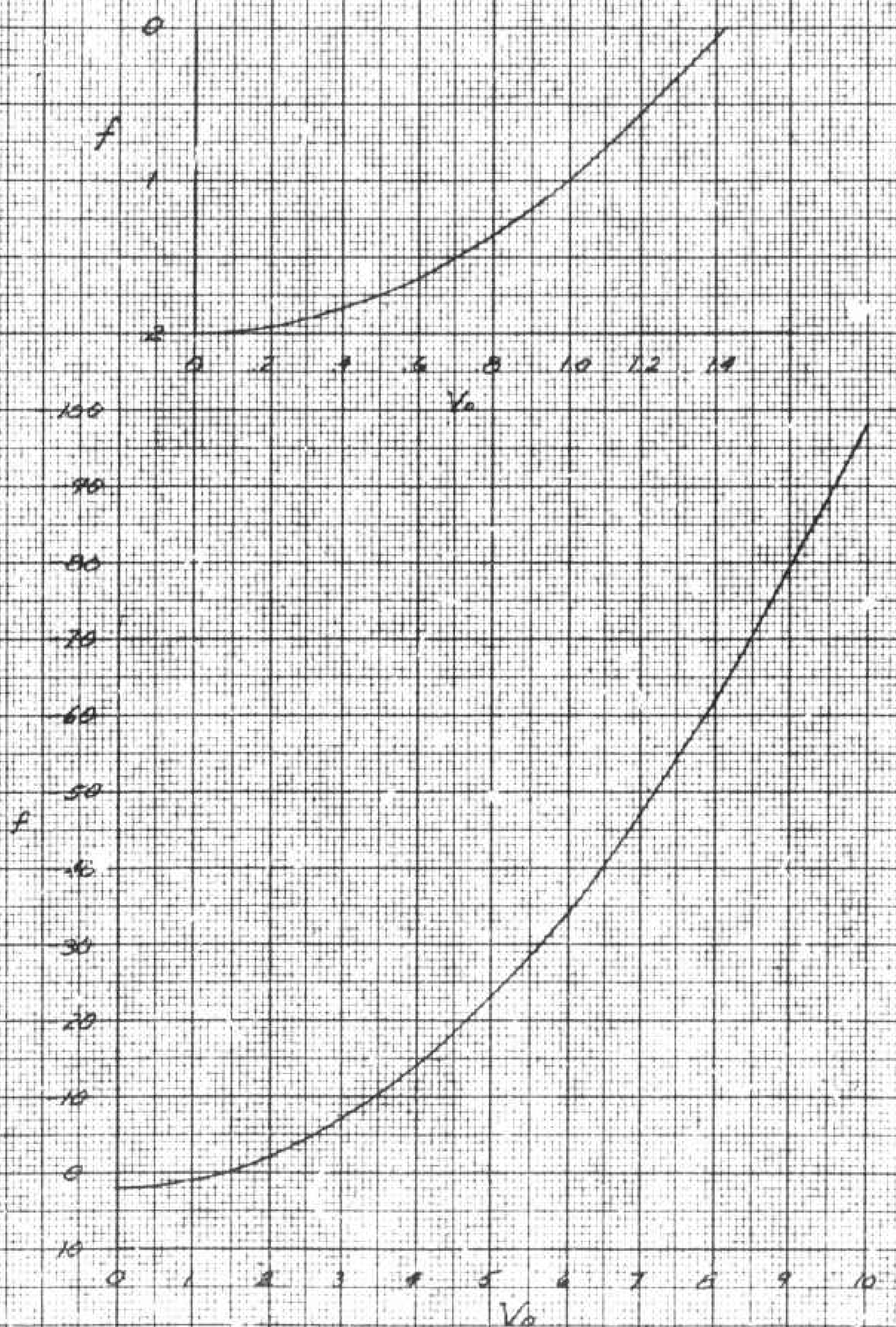


Figure 2. Velocity Parameter as a Function of the Dimensionless Velocity V_o

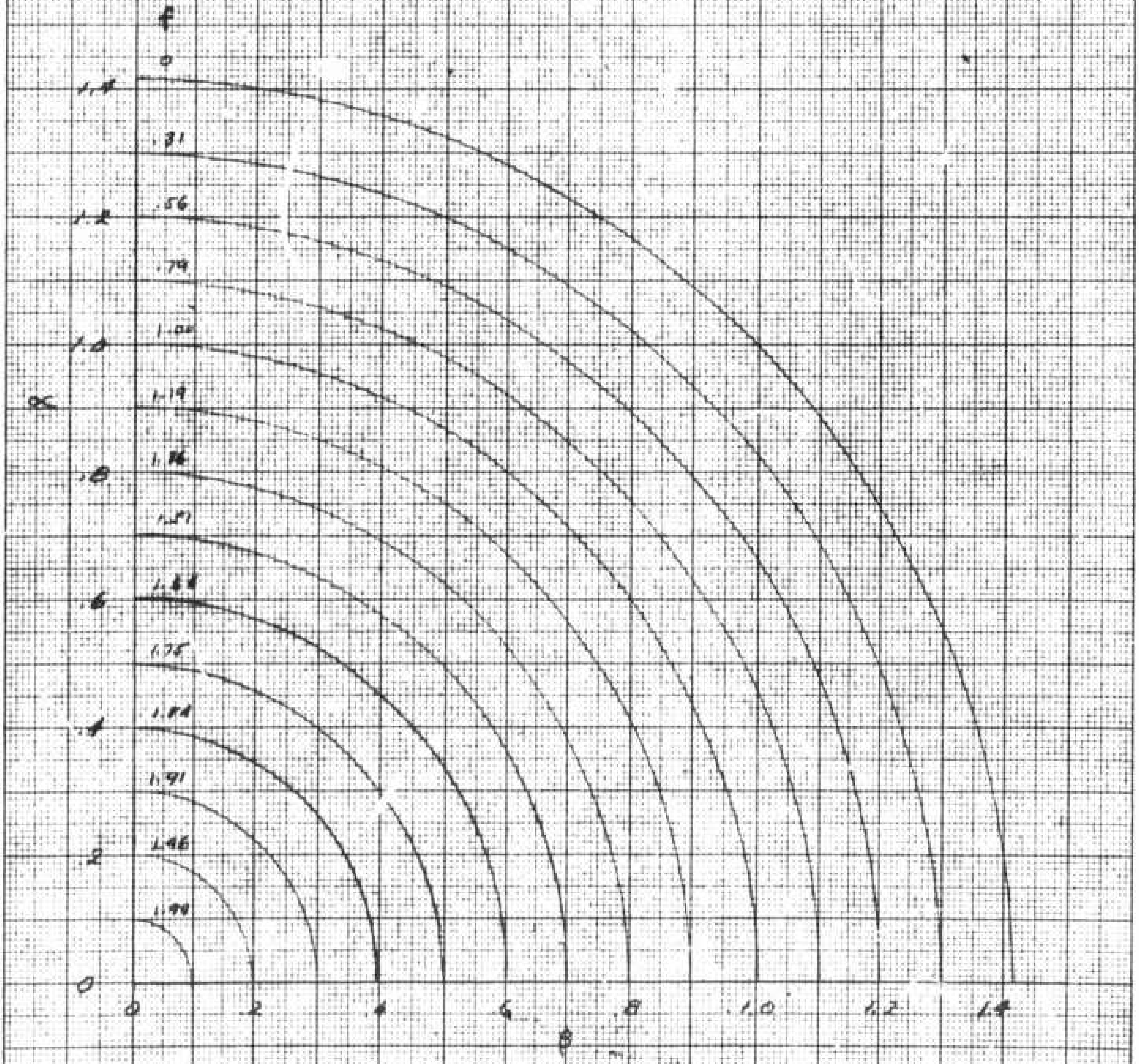


Figure 3. Velocity Parameter as a Function of α, β

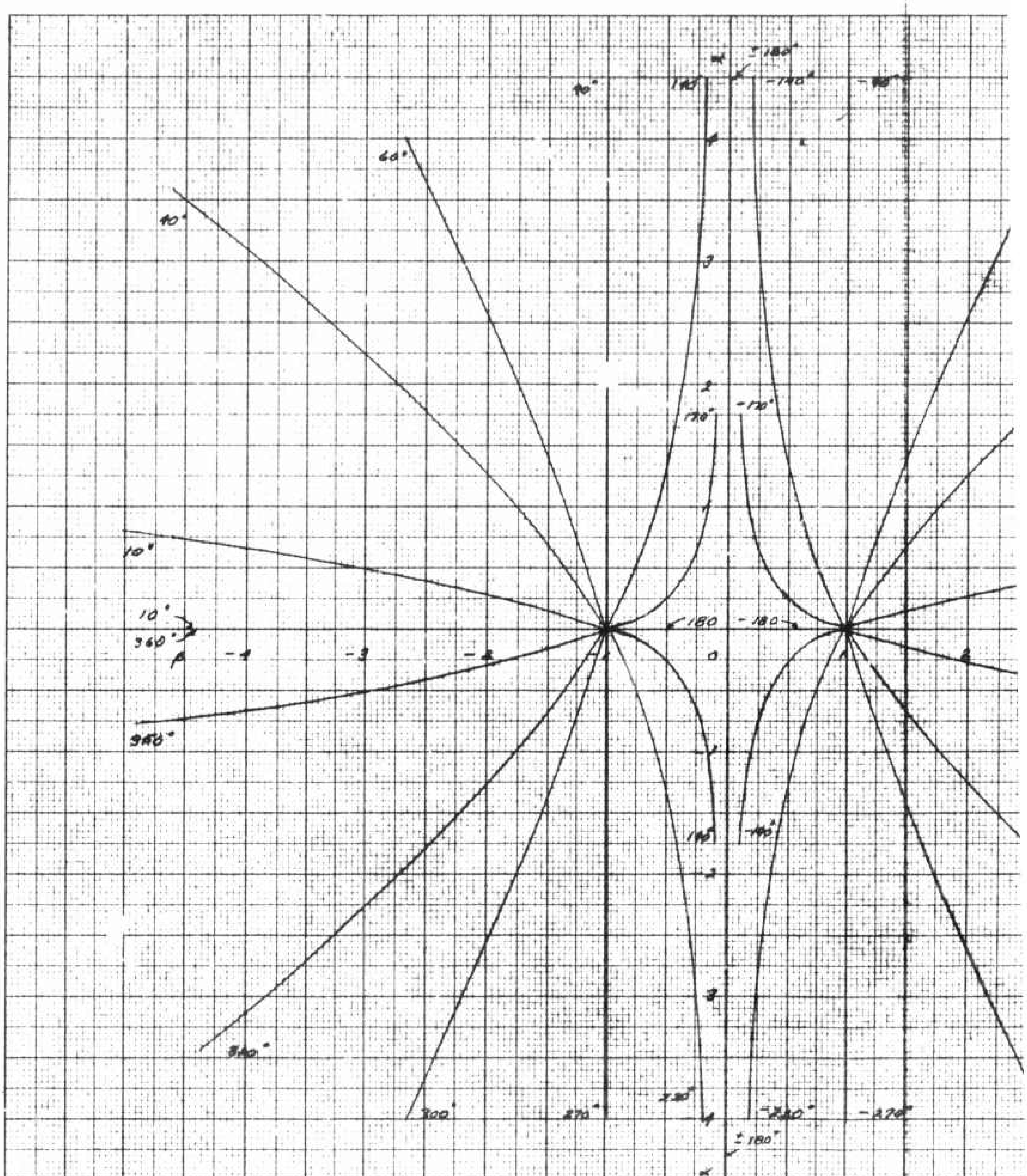
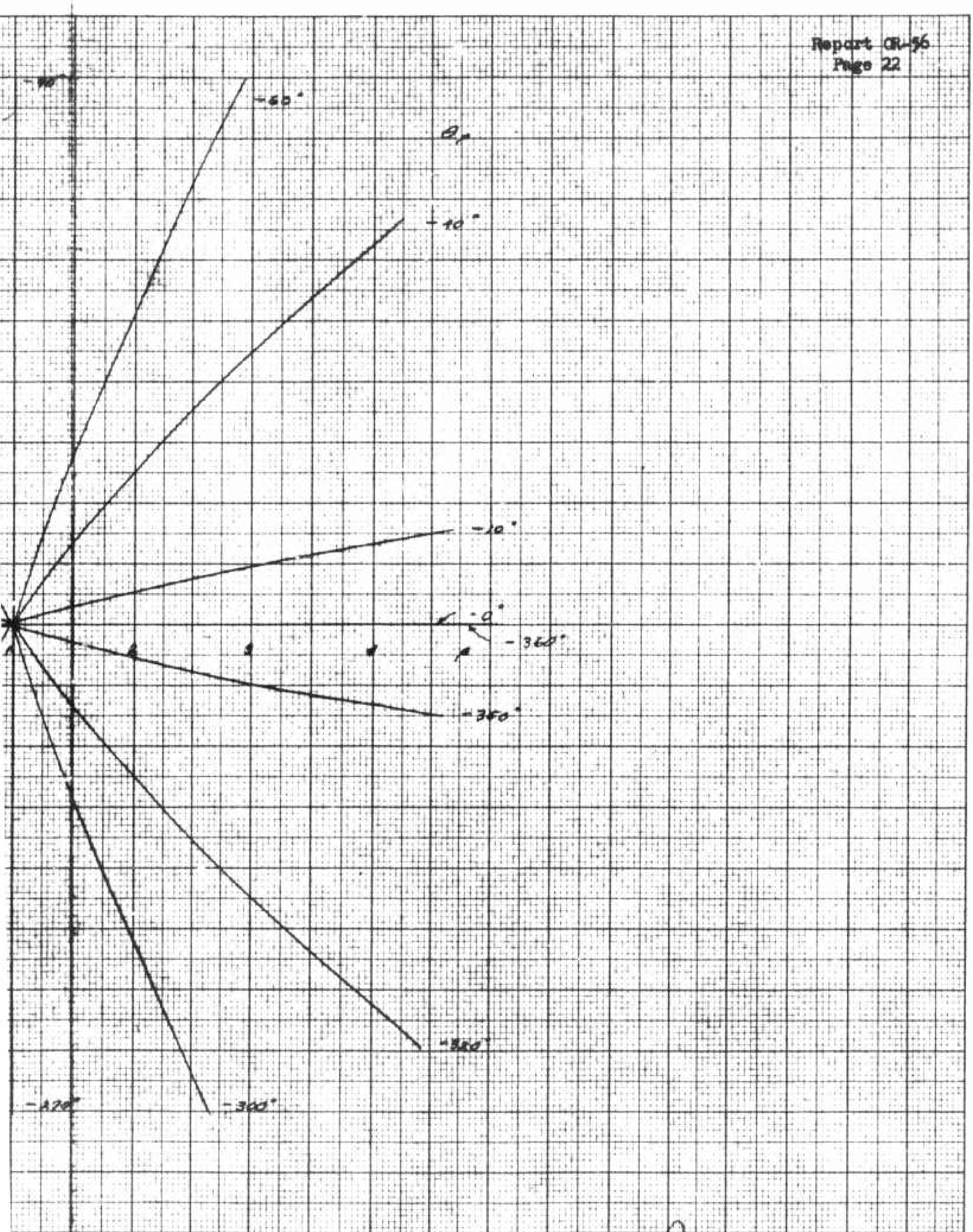


Figure 4. Perigee Angle

A



B

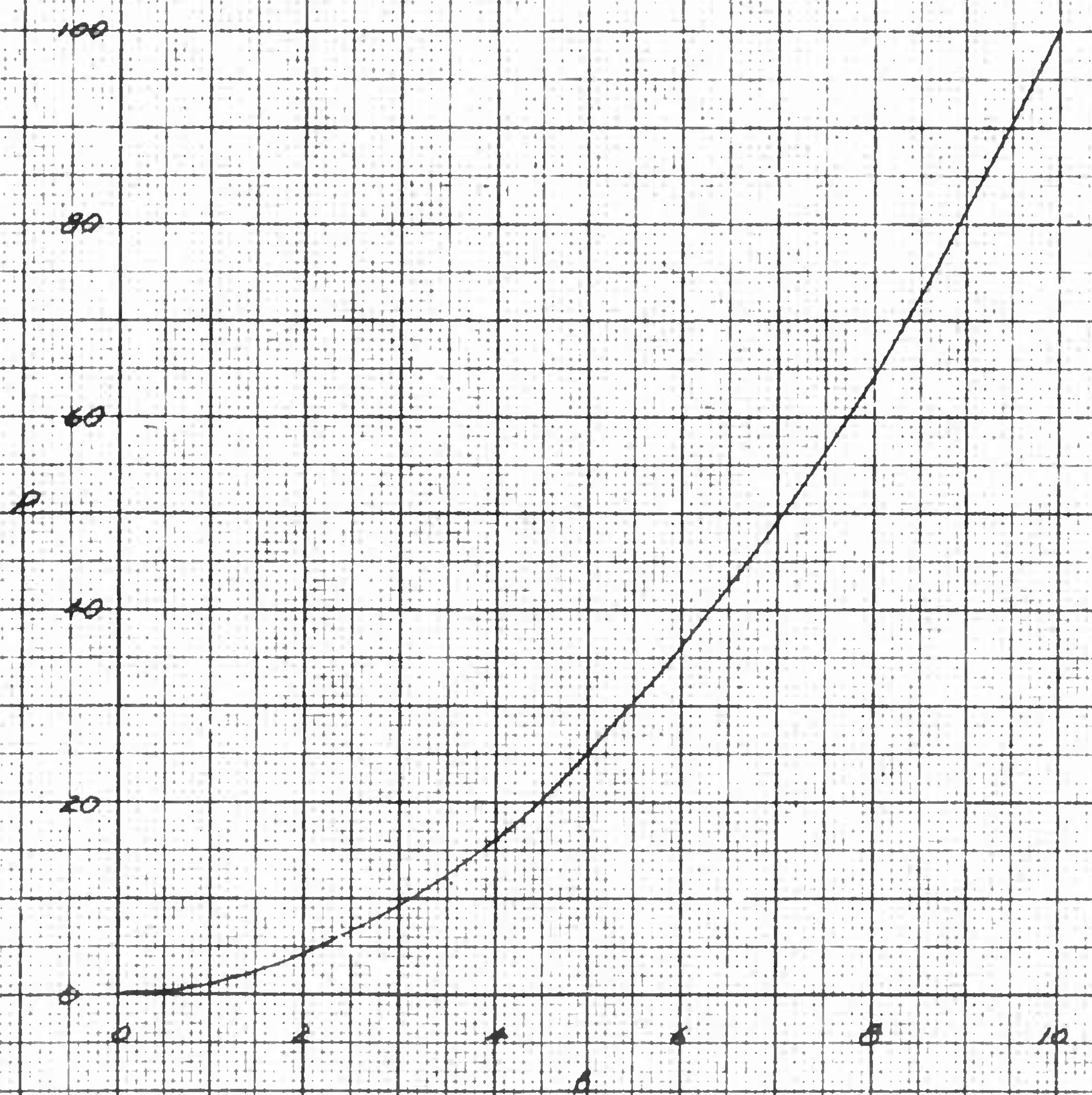


Figure 5. Semilatus Rectum

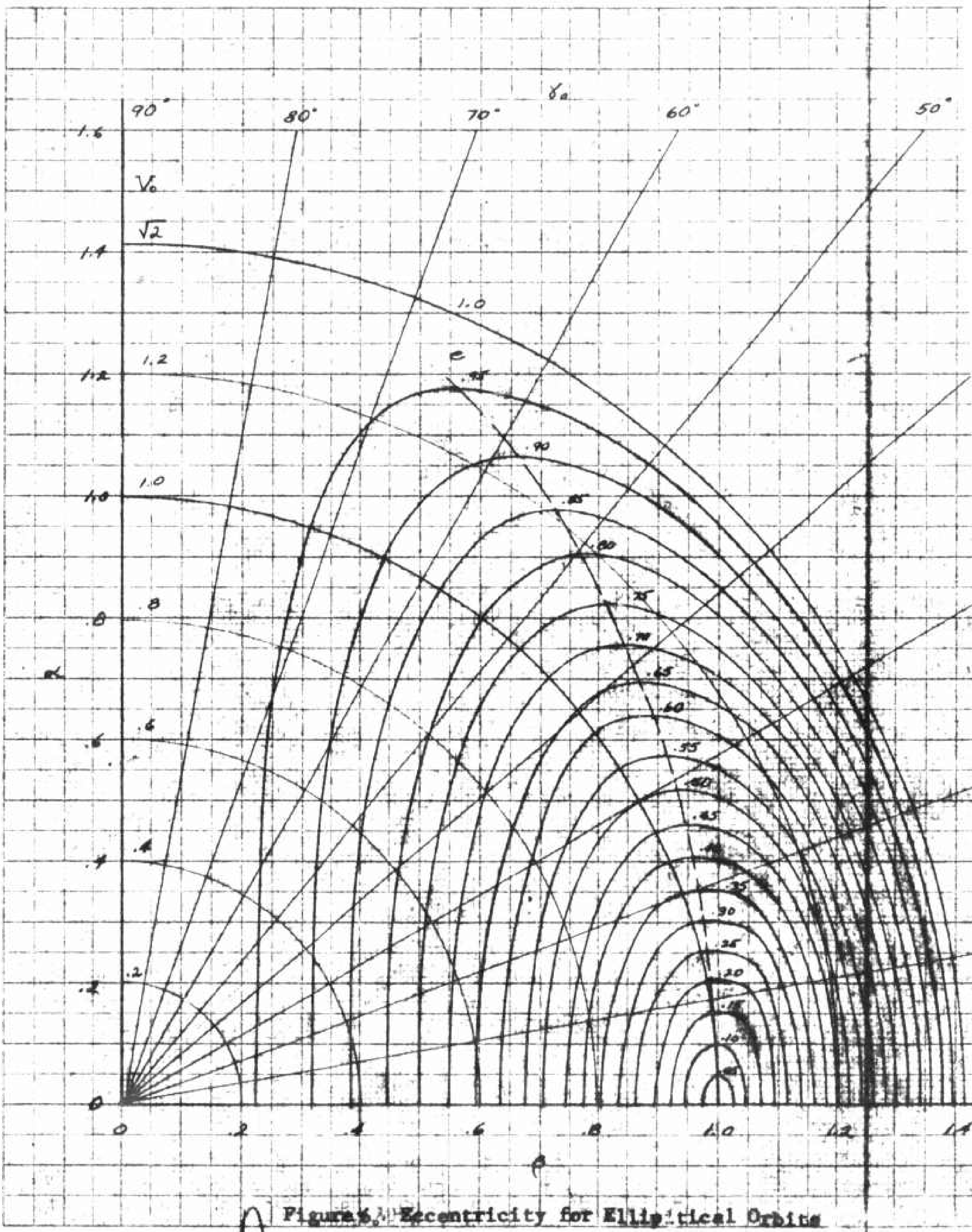
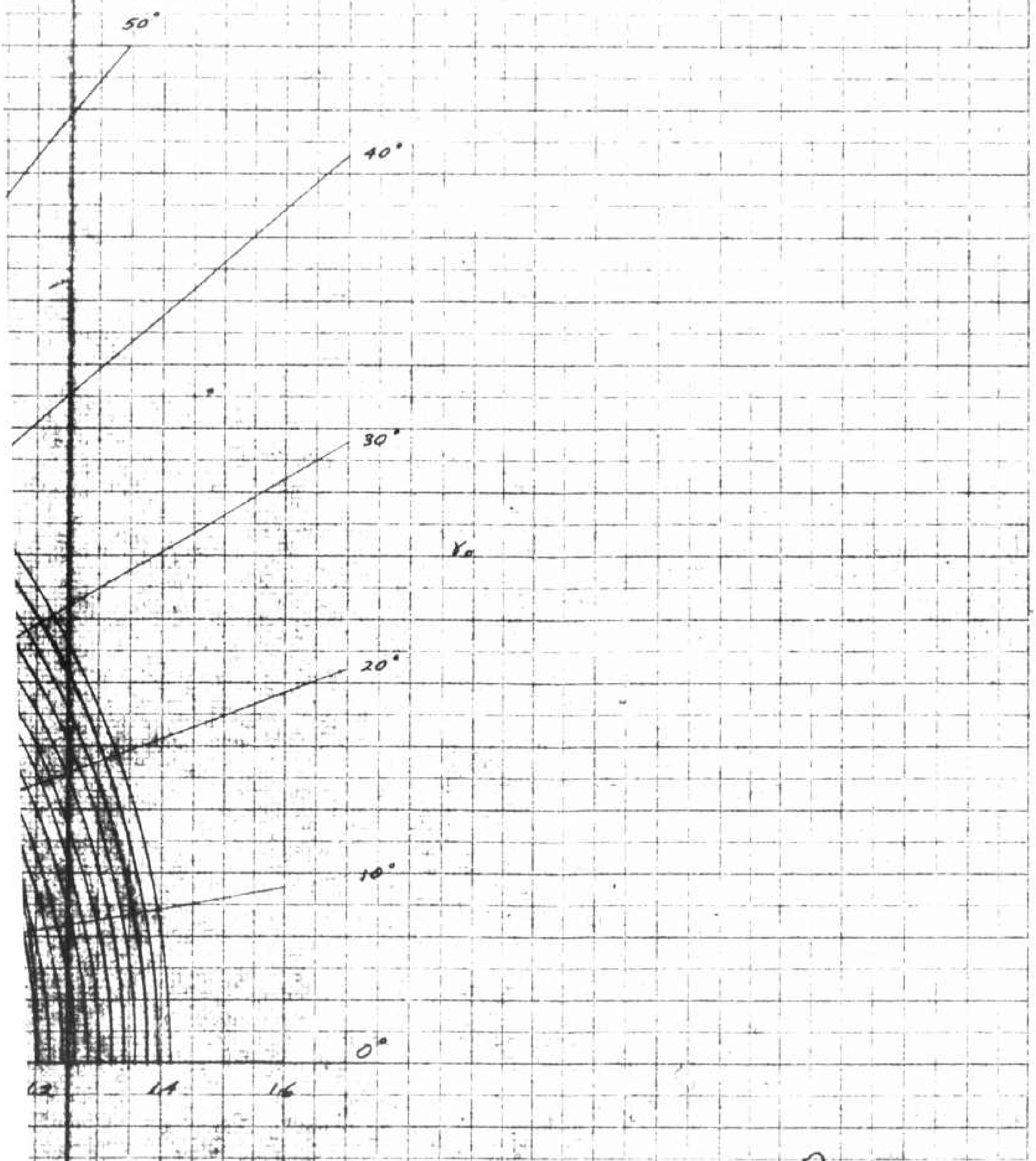


Figure 6. Eccentricity for Elliptical Orbits

A



B

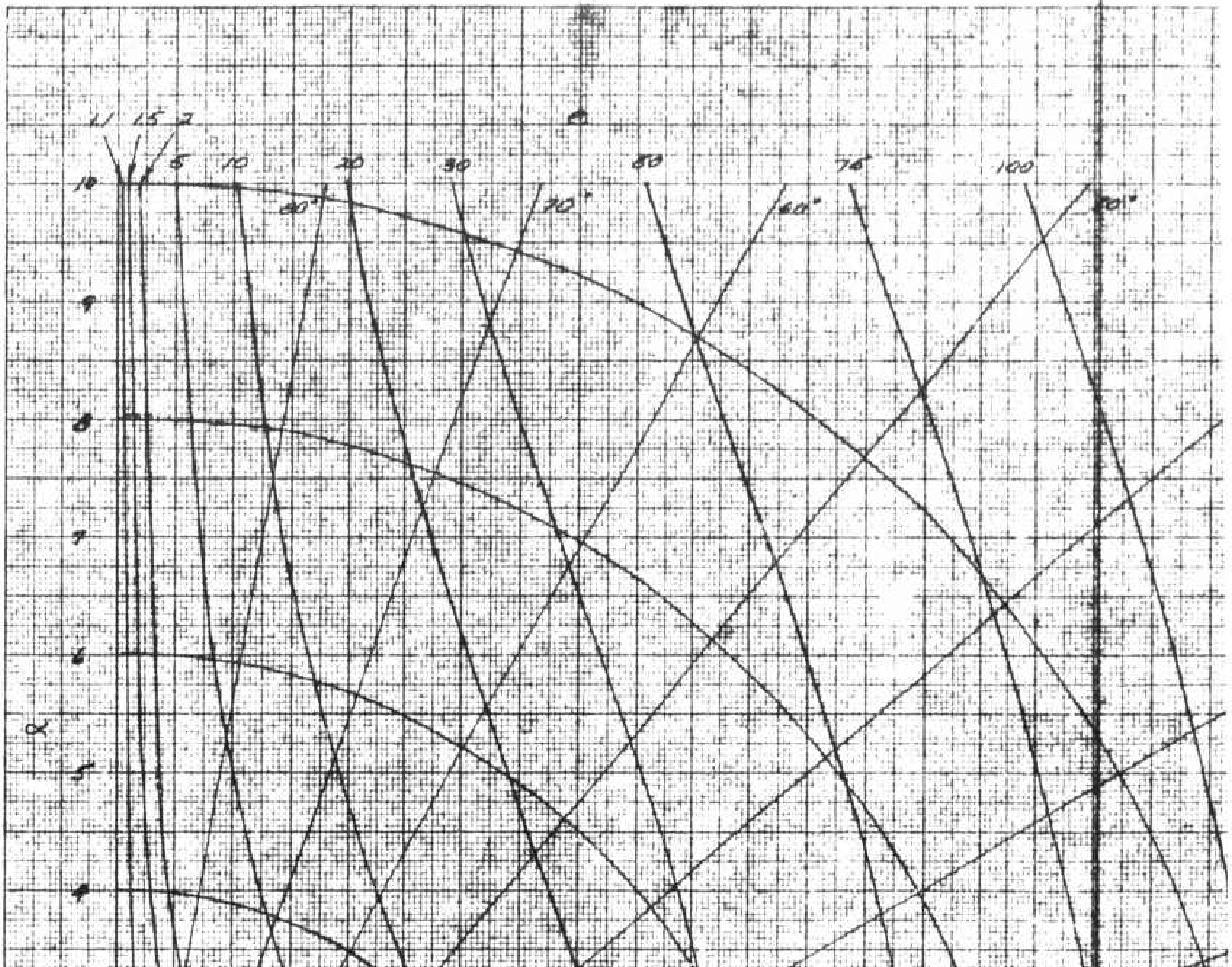
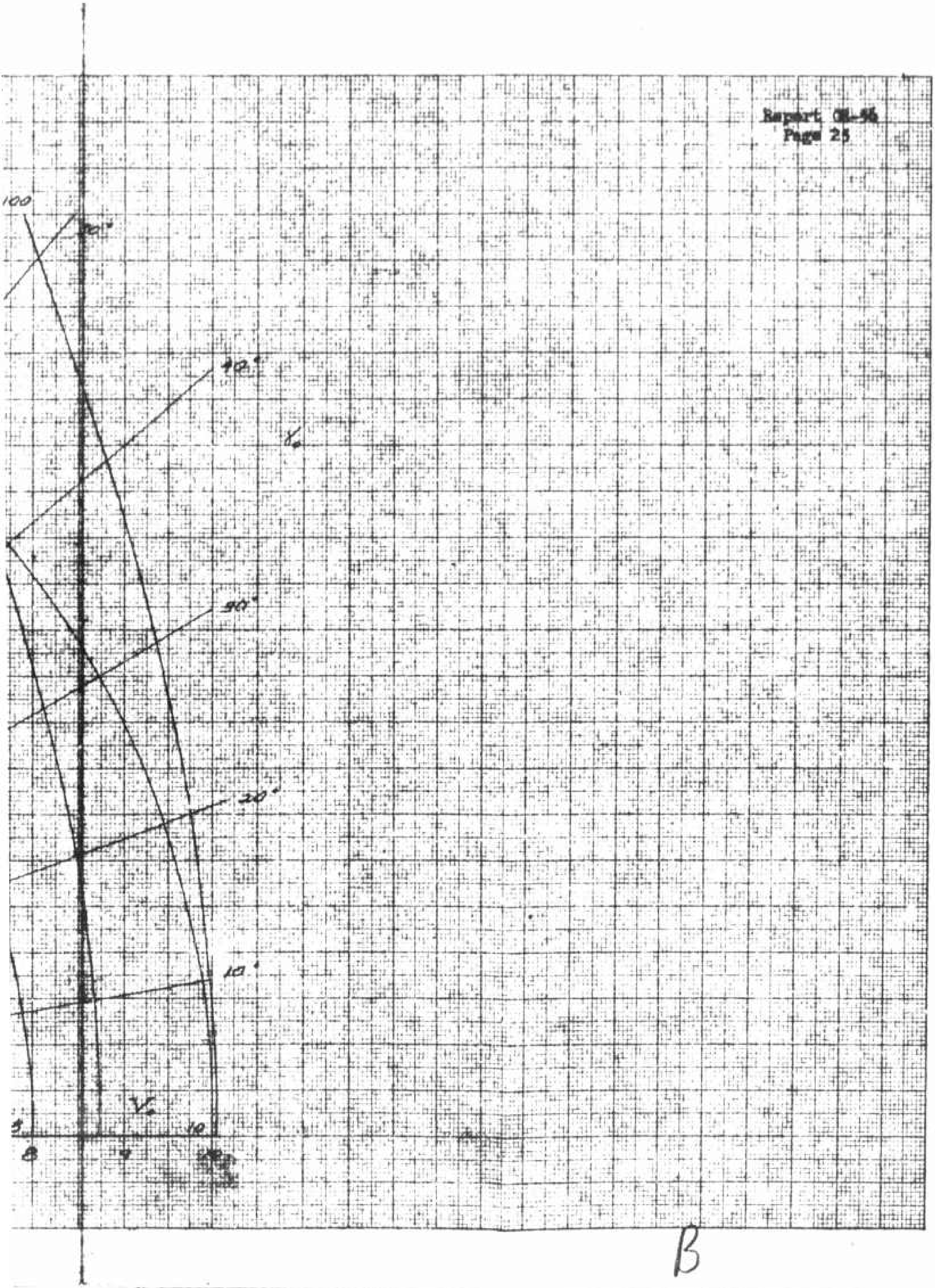


Figure 7. Eccentricity for Hyperbolic Orbits

A



B

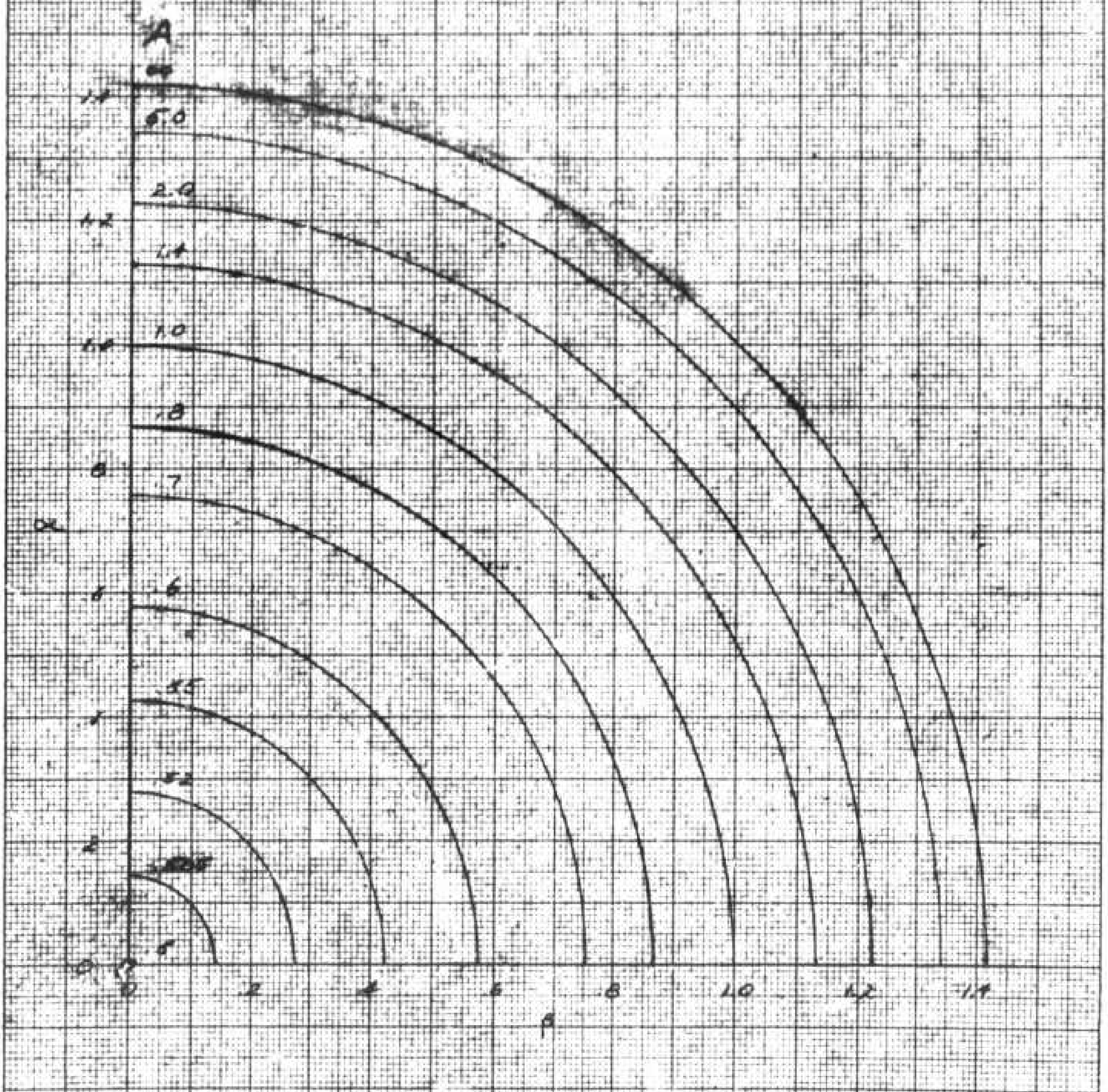


Figure [illegible]. Semimajor Axis of an Elliptical Orbit

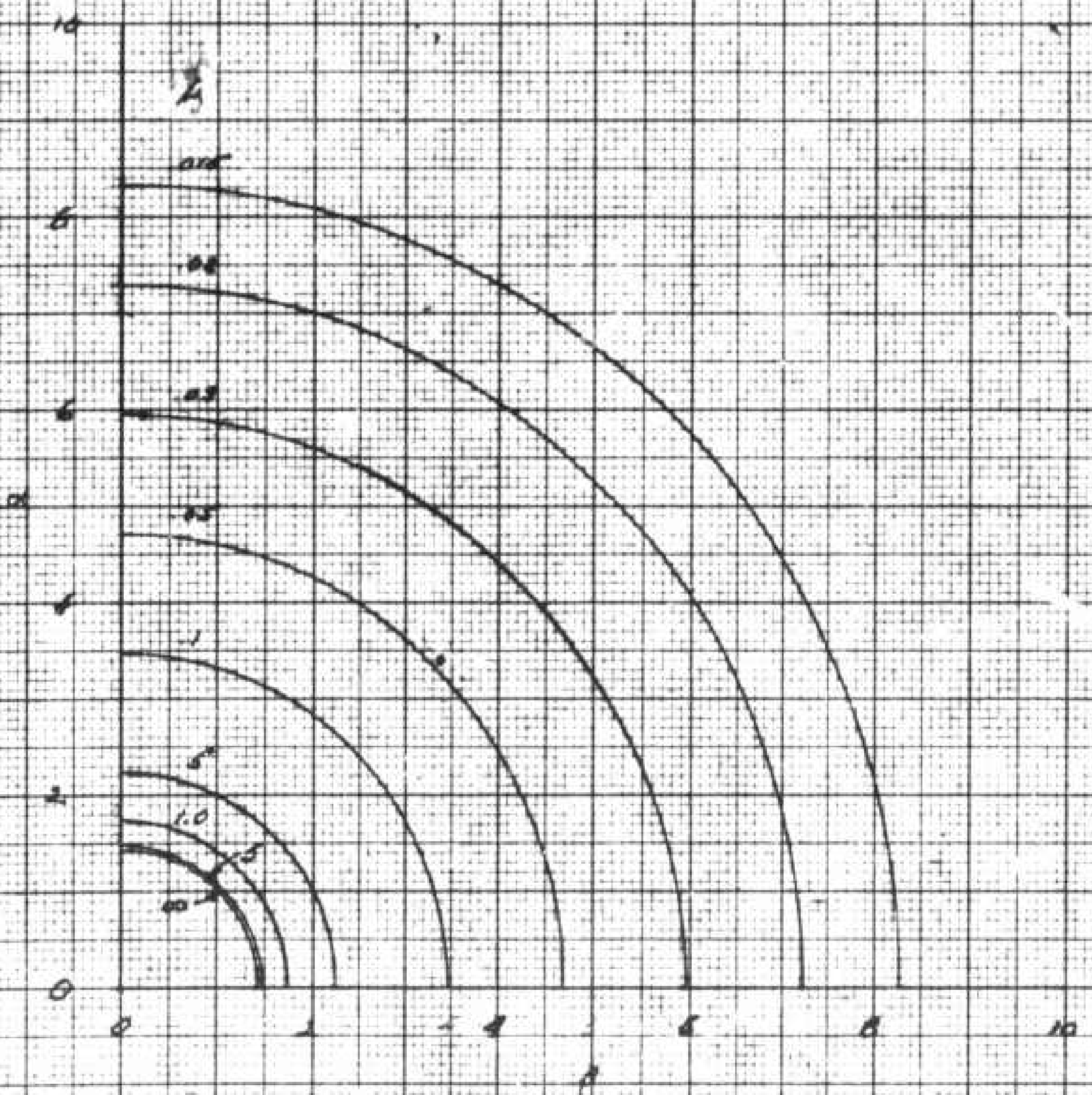


Figure 9. Semitransverse Axis of a Hyperbolic Orbit

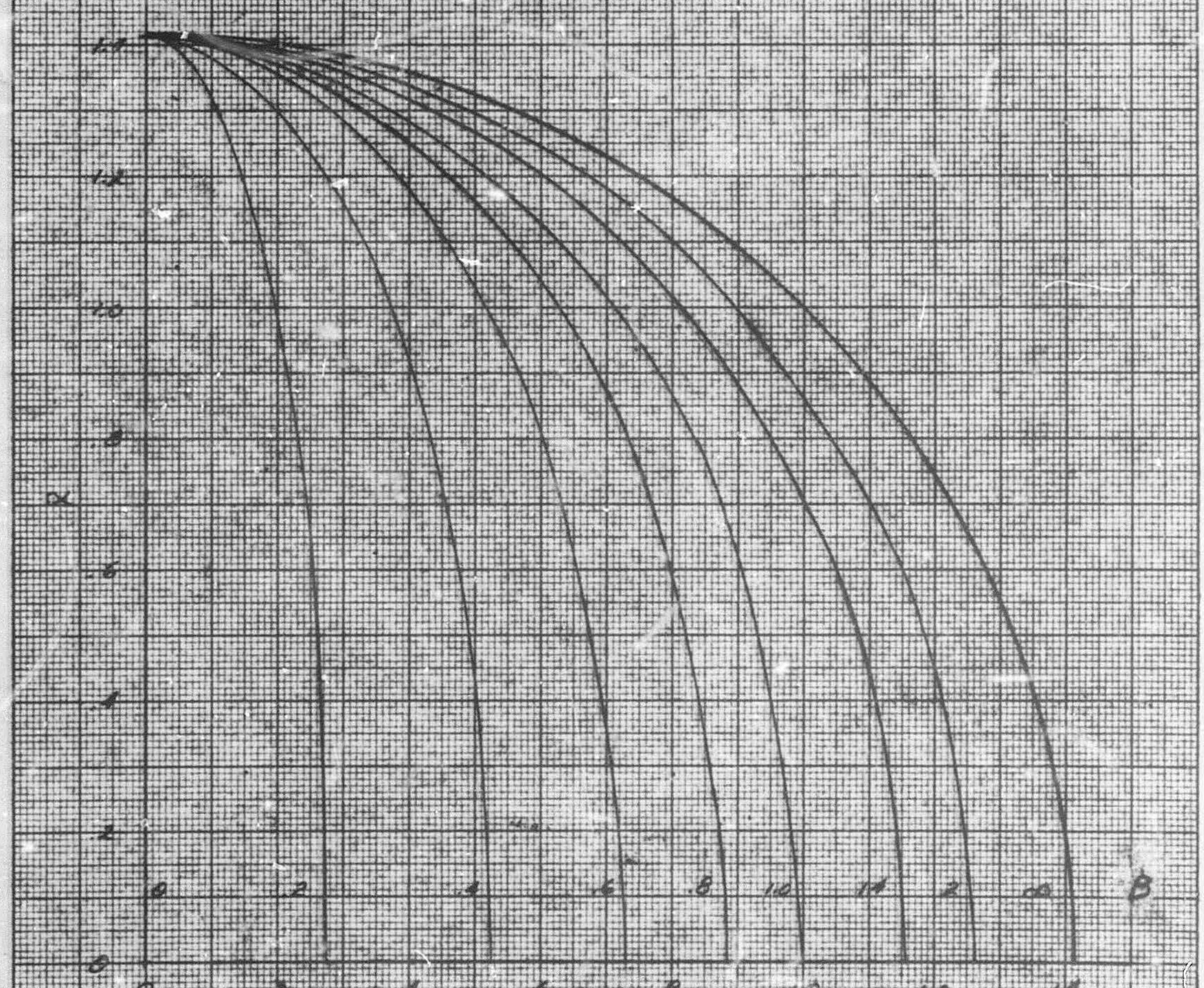


Figure 20, Semiminor Axis of an Elliptical Orbit

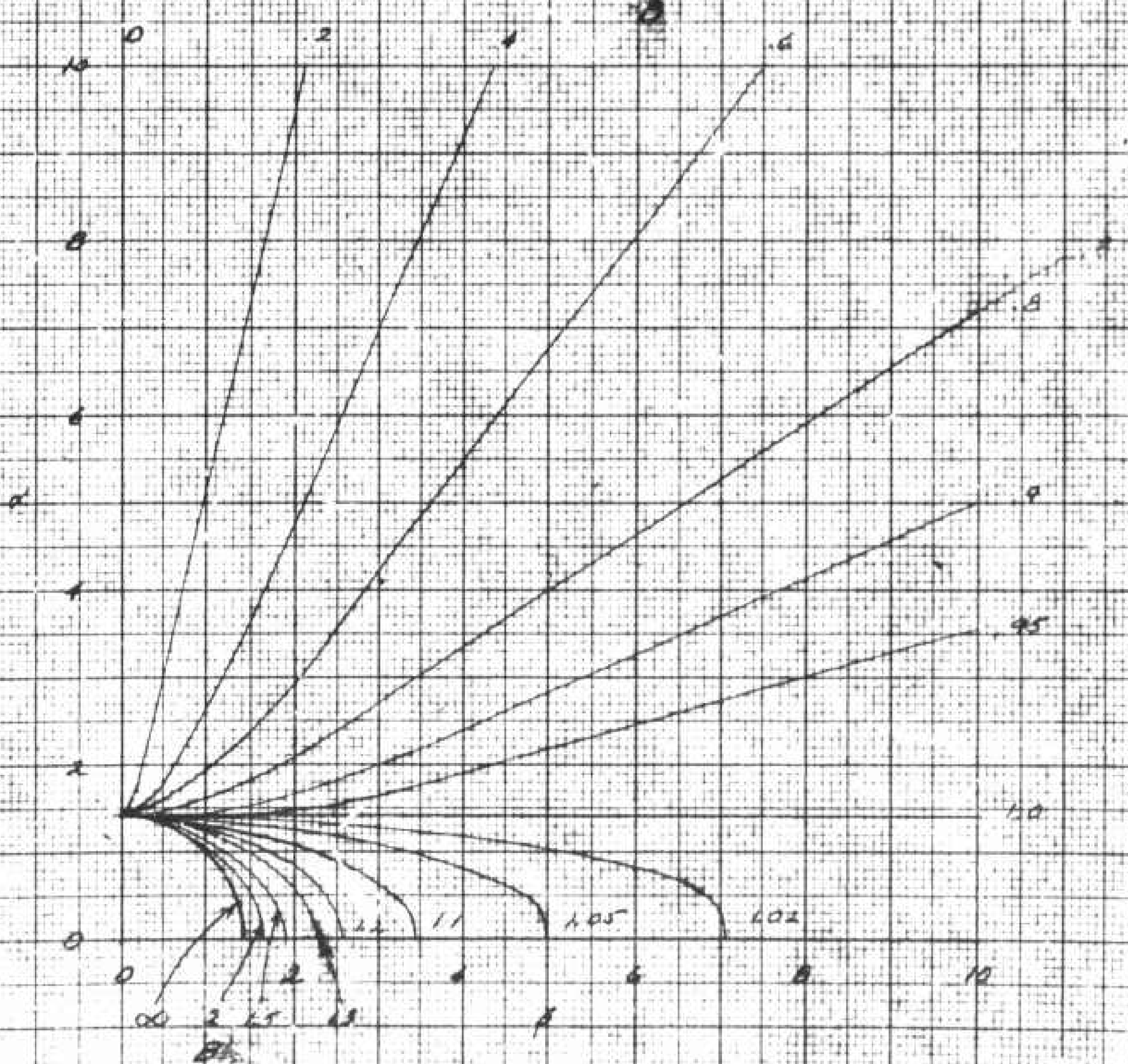


Figure 11. Semiconjugate axis of Hyperbolic Orbit

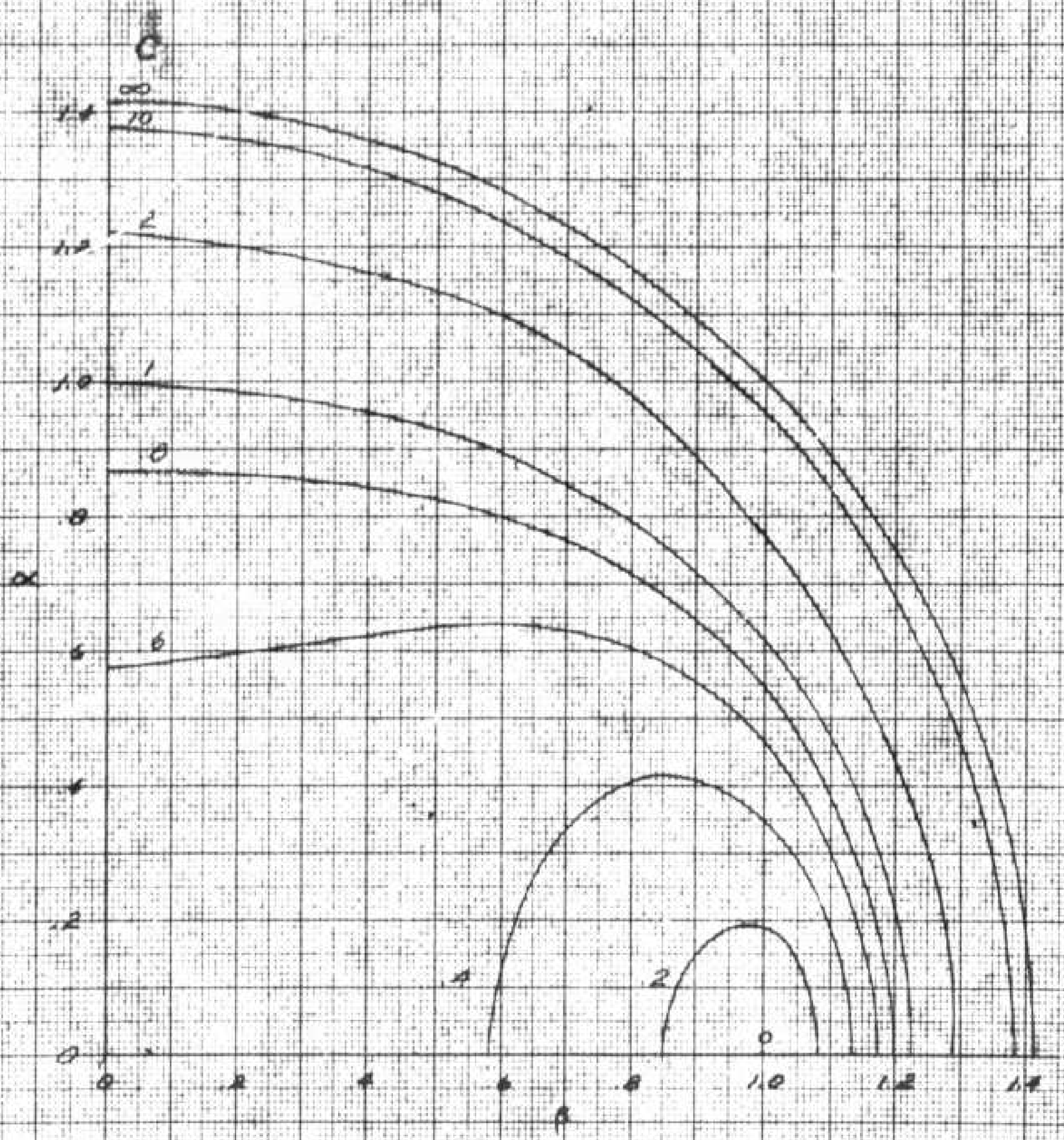


Figure 12. Linear Eccentricity of an Elliptical Orbit

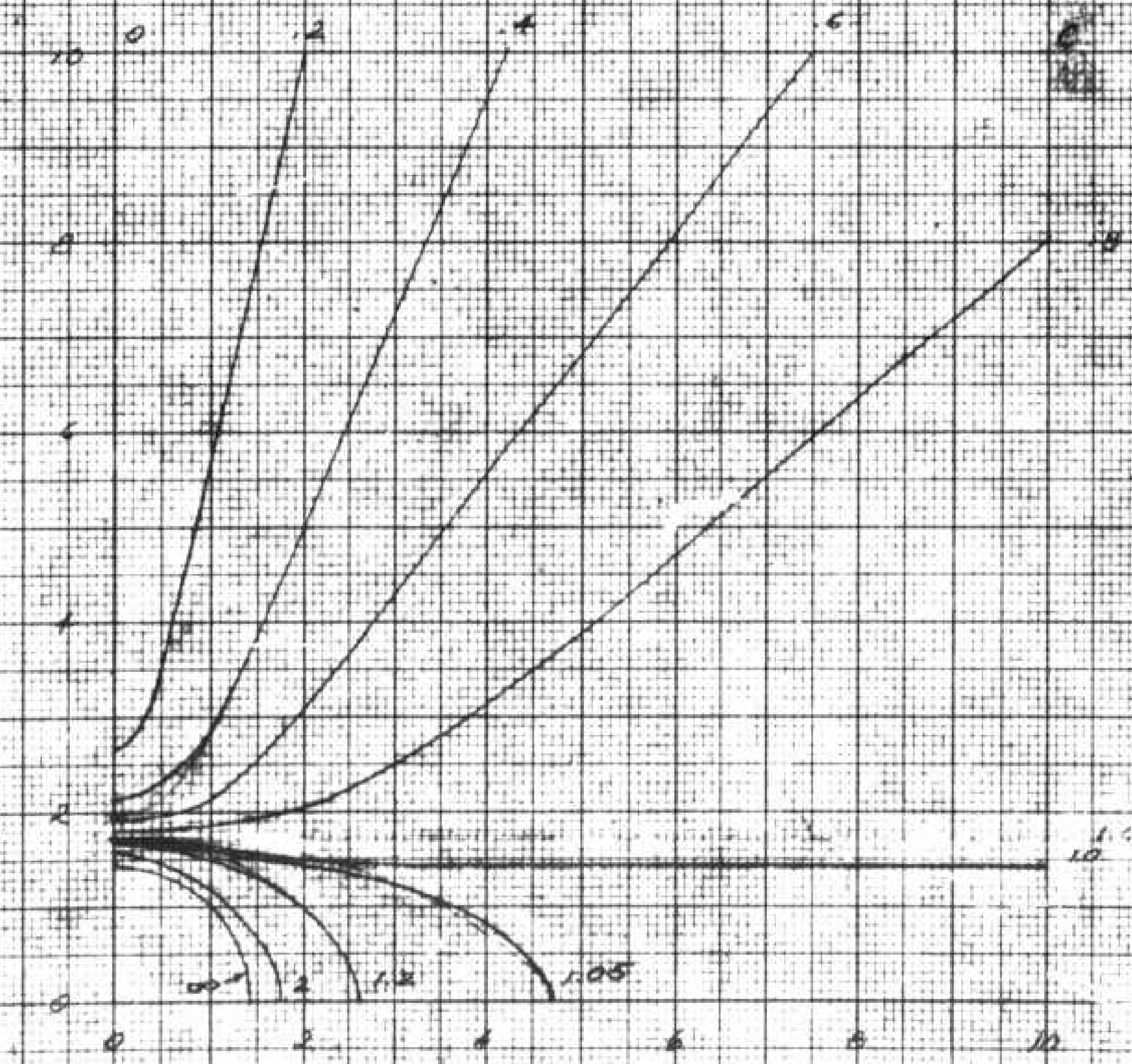


Figure 13. Linear Eccentricity of a Hyperbolic Orbit

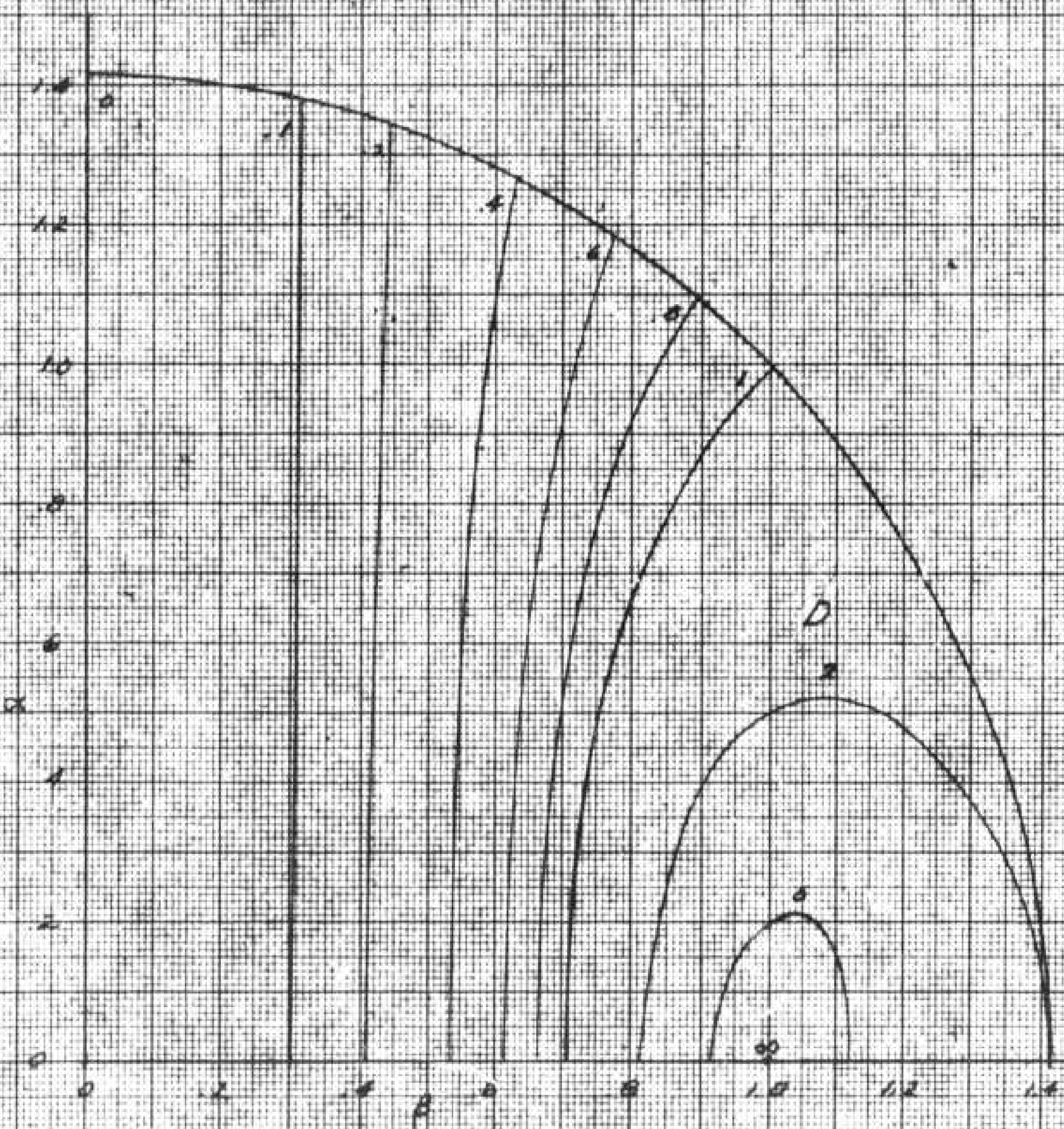


Figure 16. Directrix Distance for an Elliptical Orbit

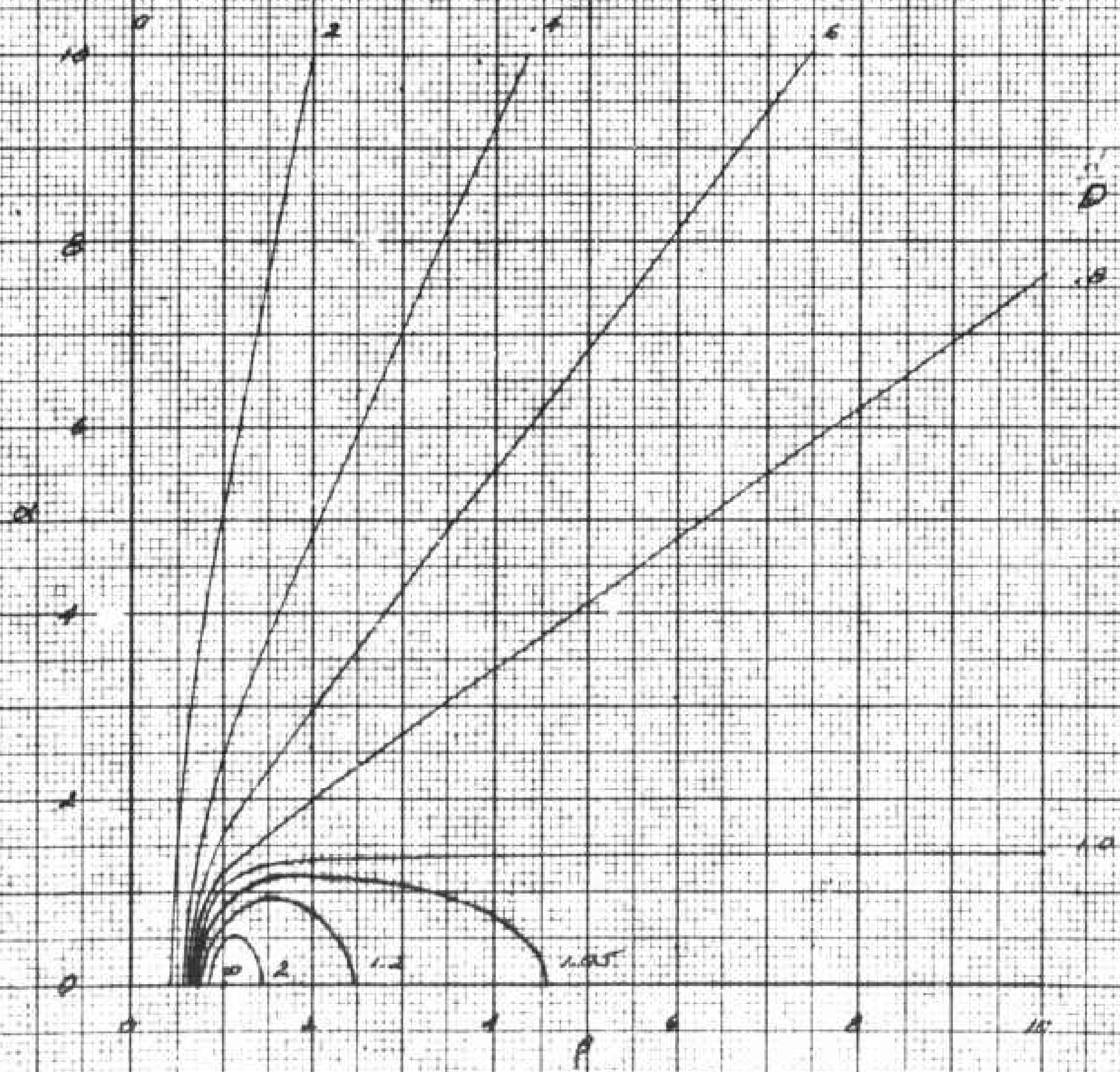


Figure 14. Directrix Distance for a Hyperbolic Orbit

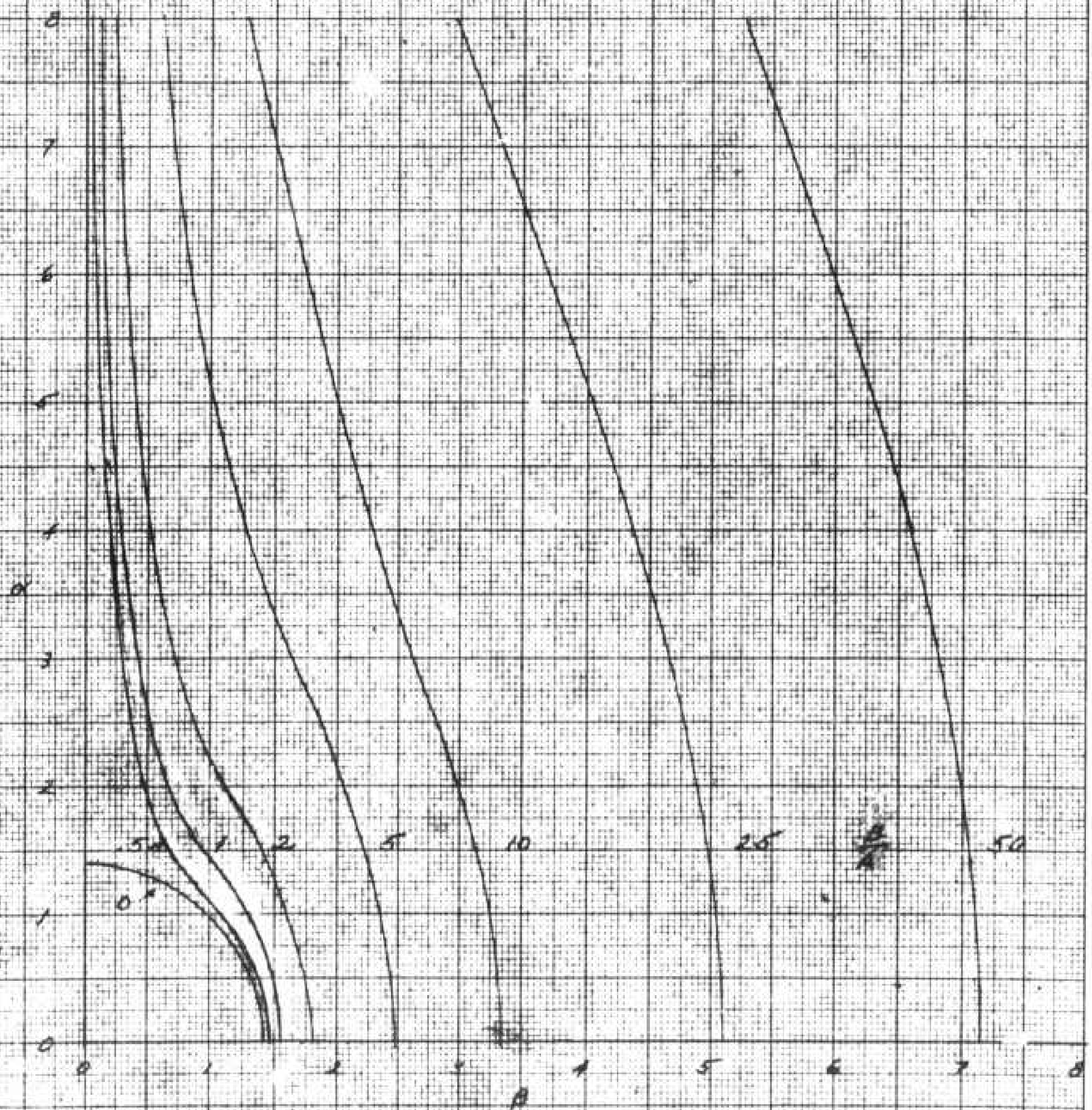


Figure 16. Asymptotes of a Hyperbolic Orbit

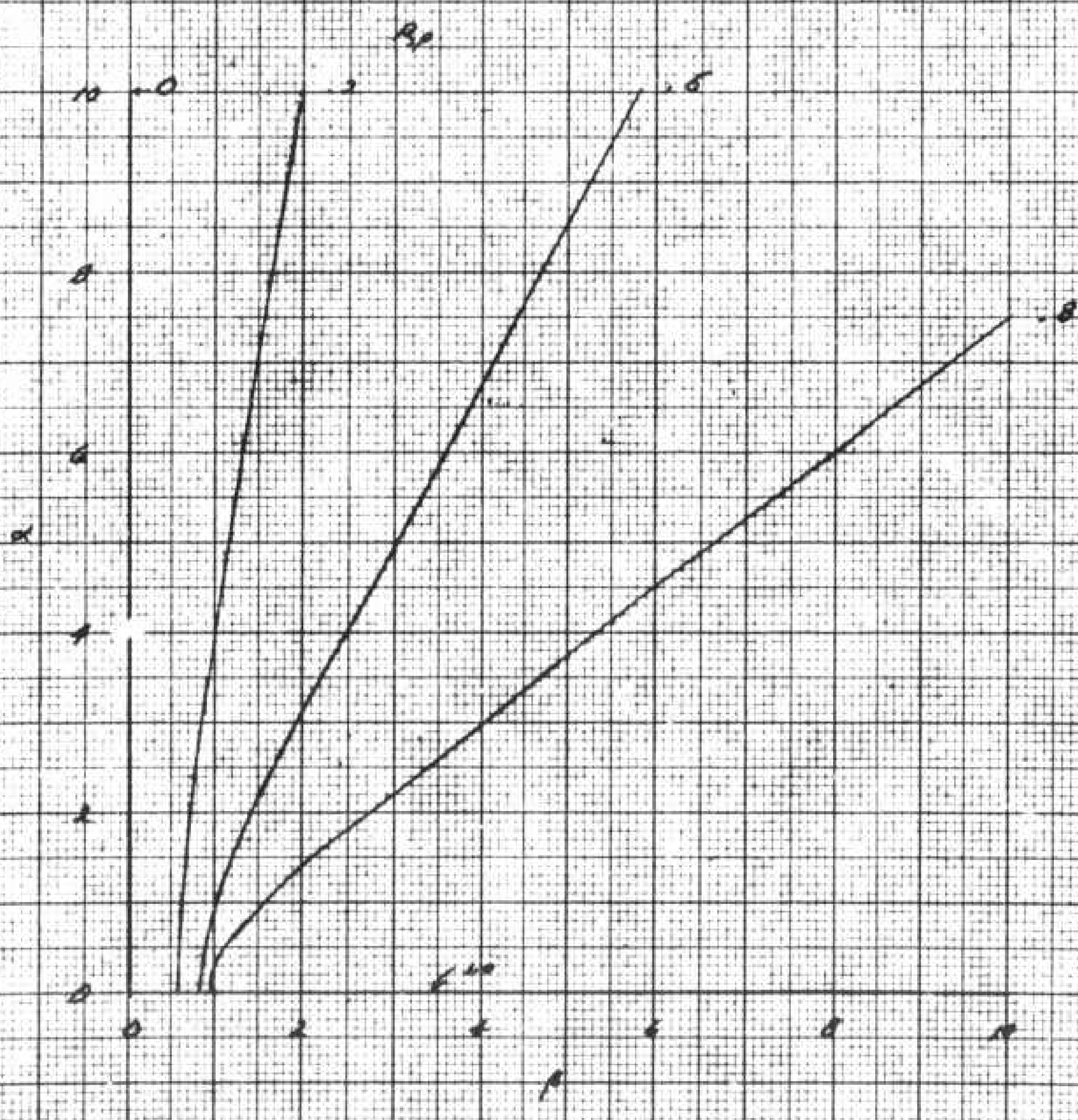


Figure 17. Distance at Perigee

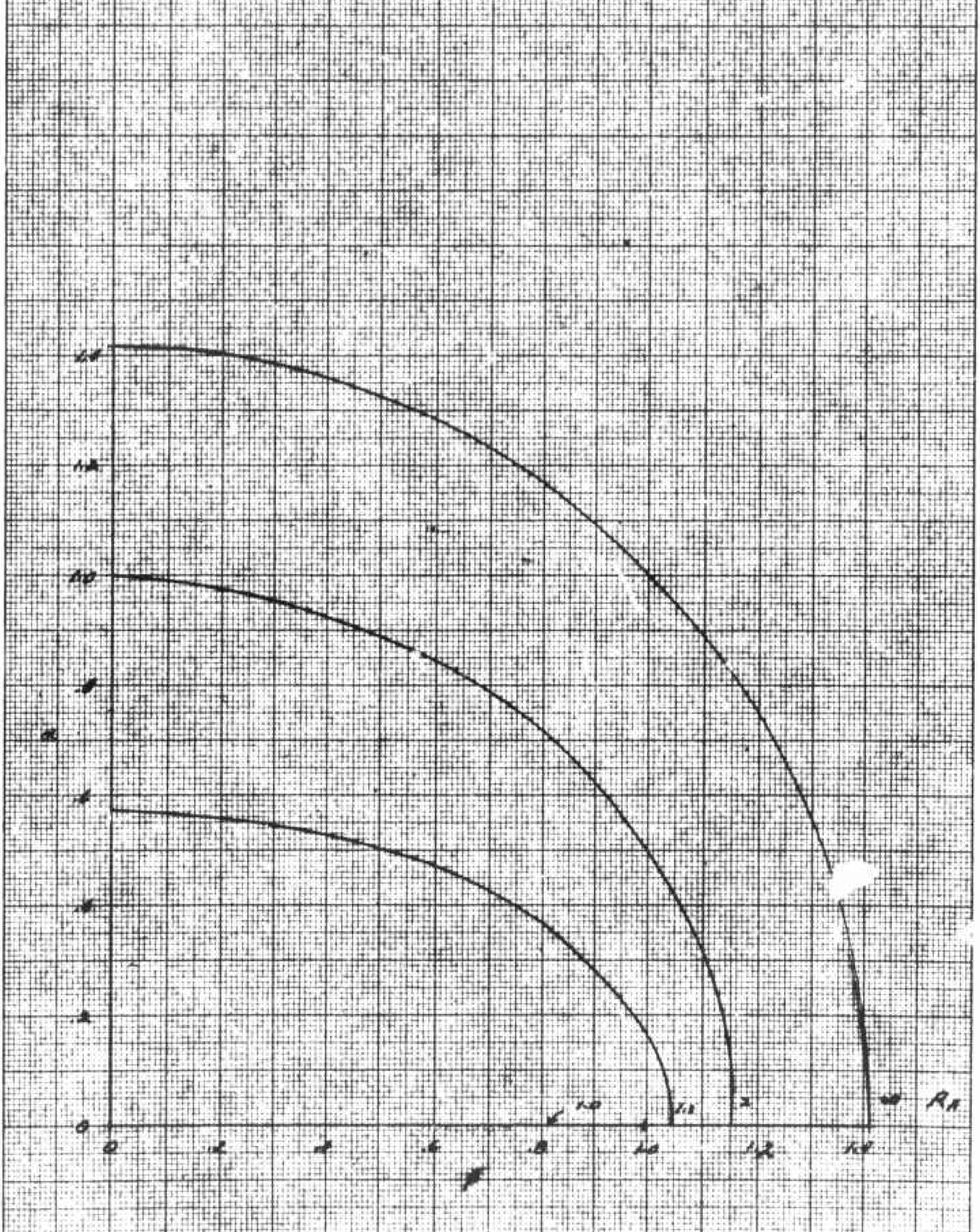
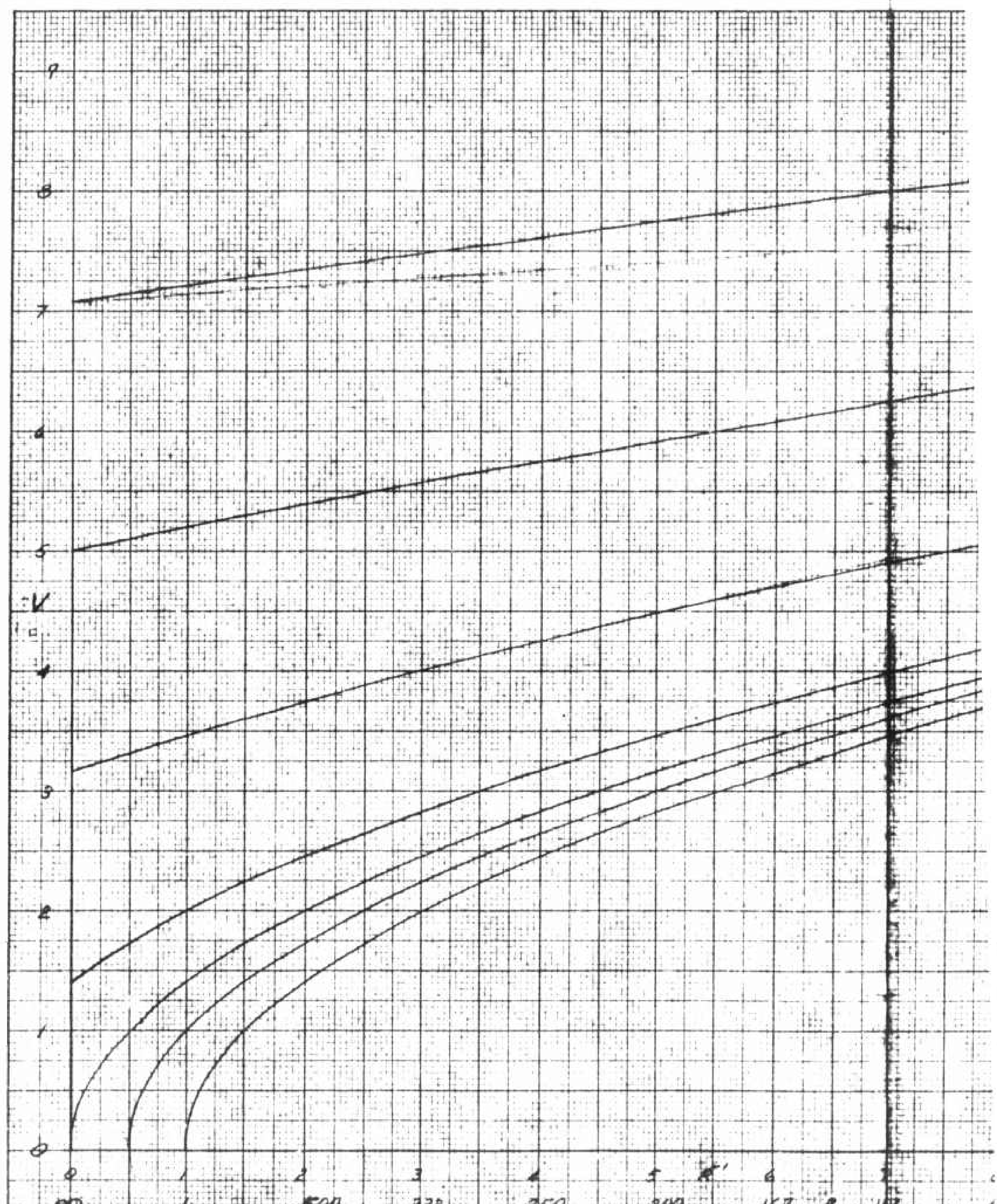
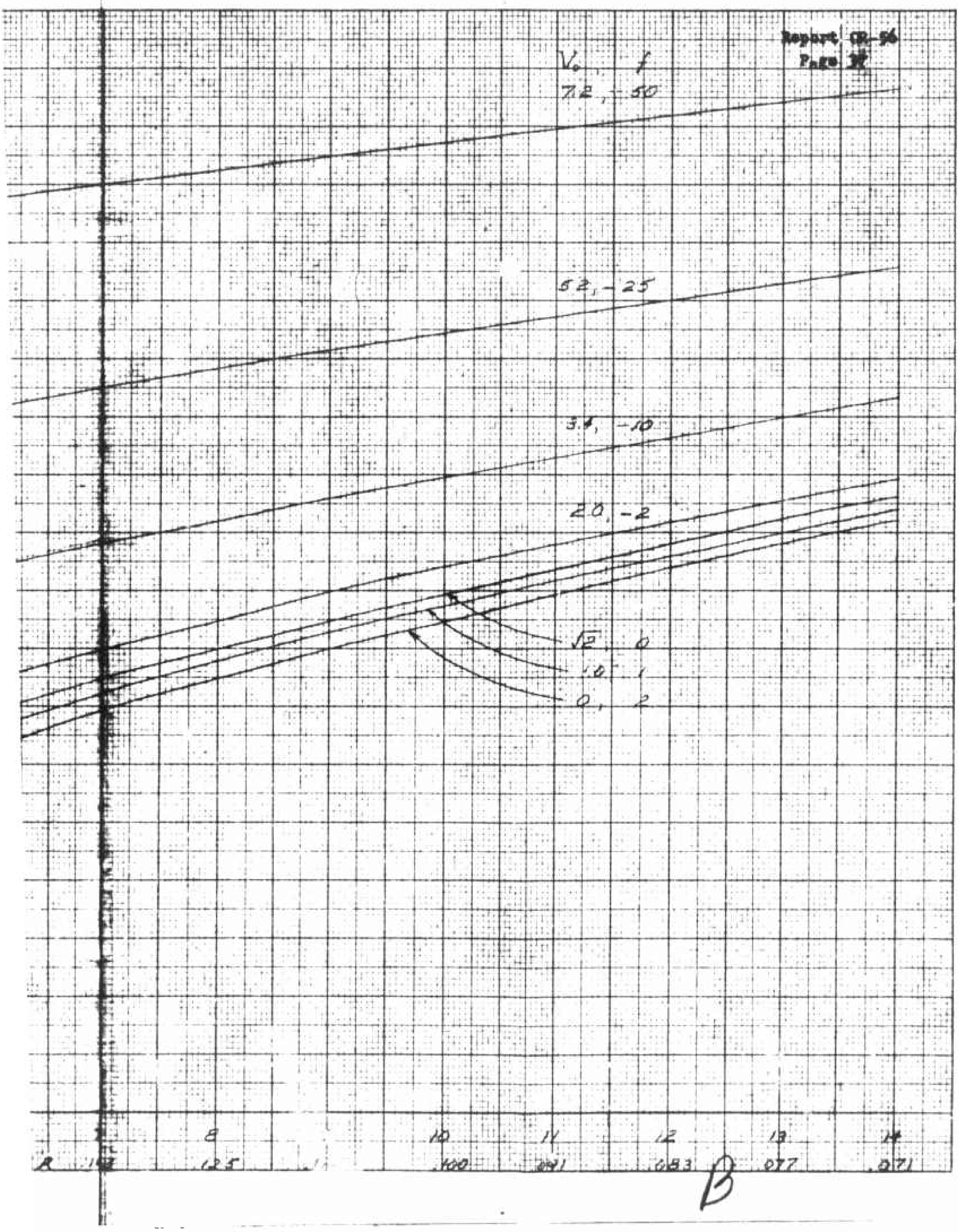


Figure 15. Distance at Apogee



A Figure 10. Velicity as a Function of Distance


V₀, f
7.2, -50
5.2, -25
3.4, -10
2.0, -2
√2, 0
1.0, 1
0, 2
8
10
11
12
13
14
R
.125
.100
.091
.083
.077
.071
β

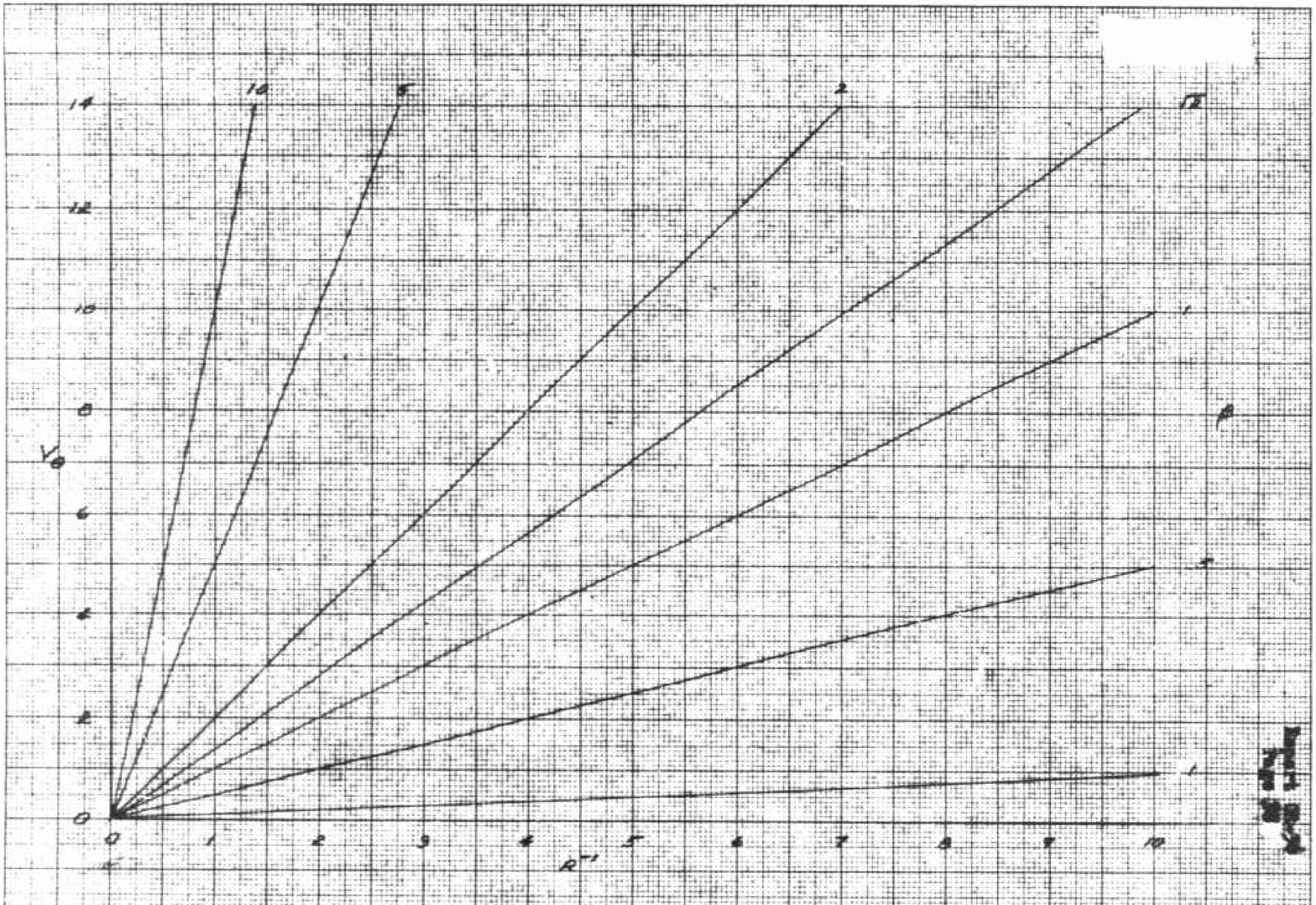


Figure 20. Normal Velocity

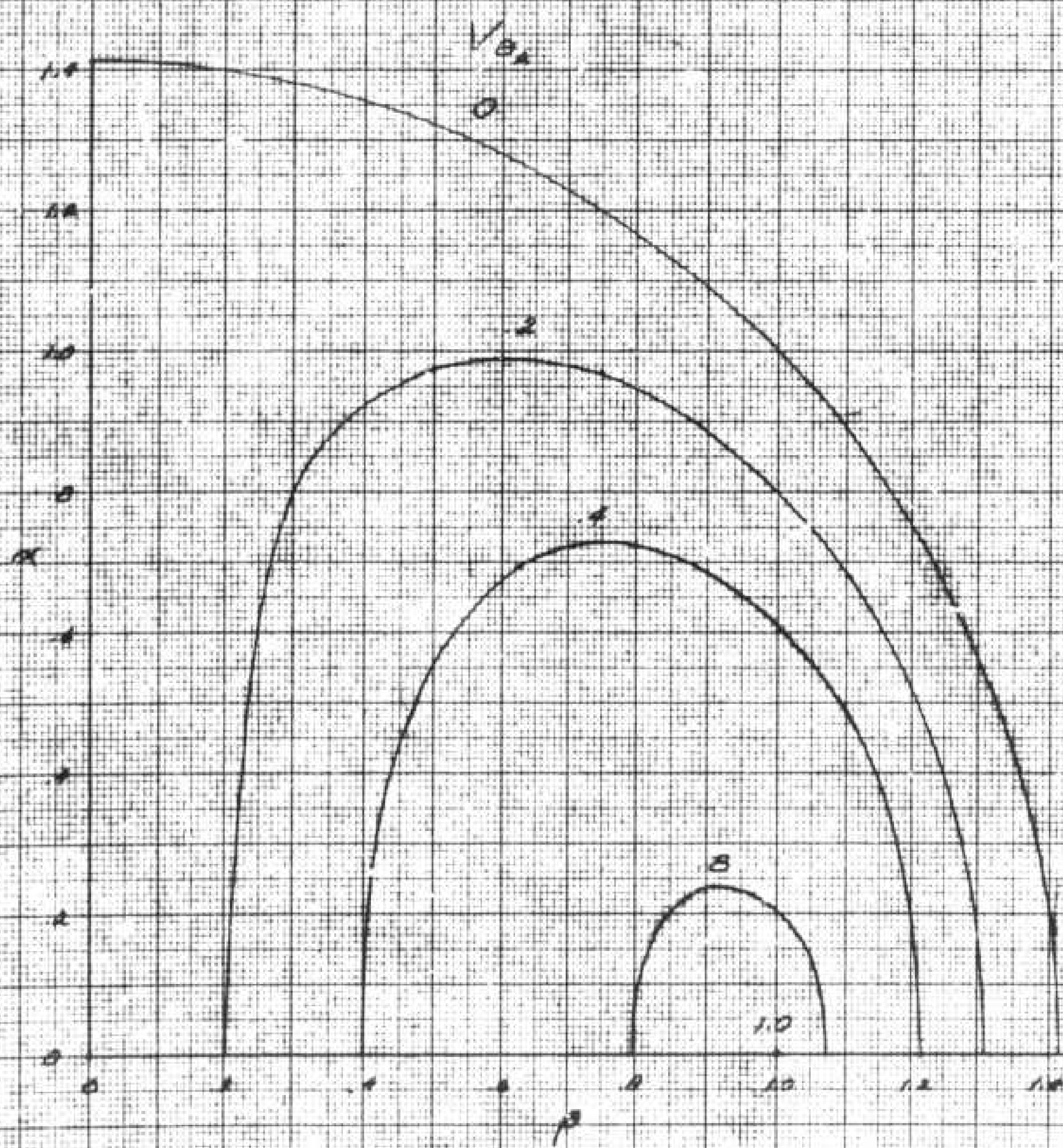


Figure 21. Apogee Velocity

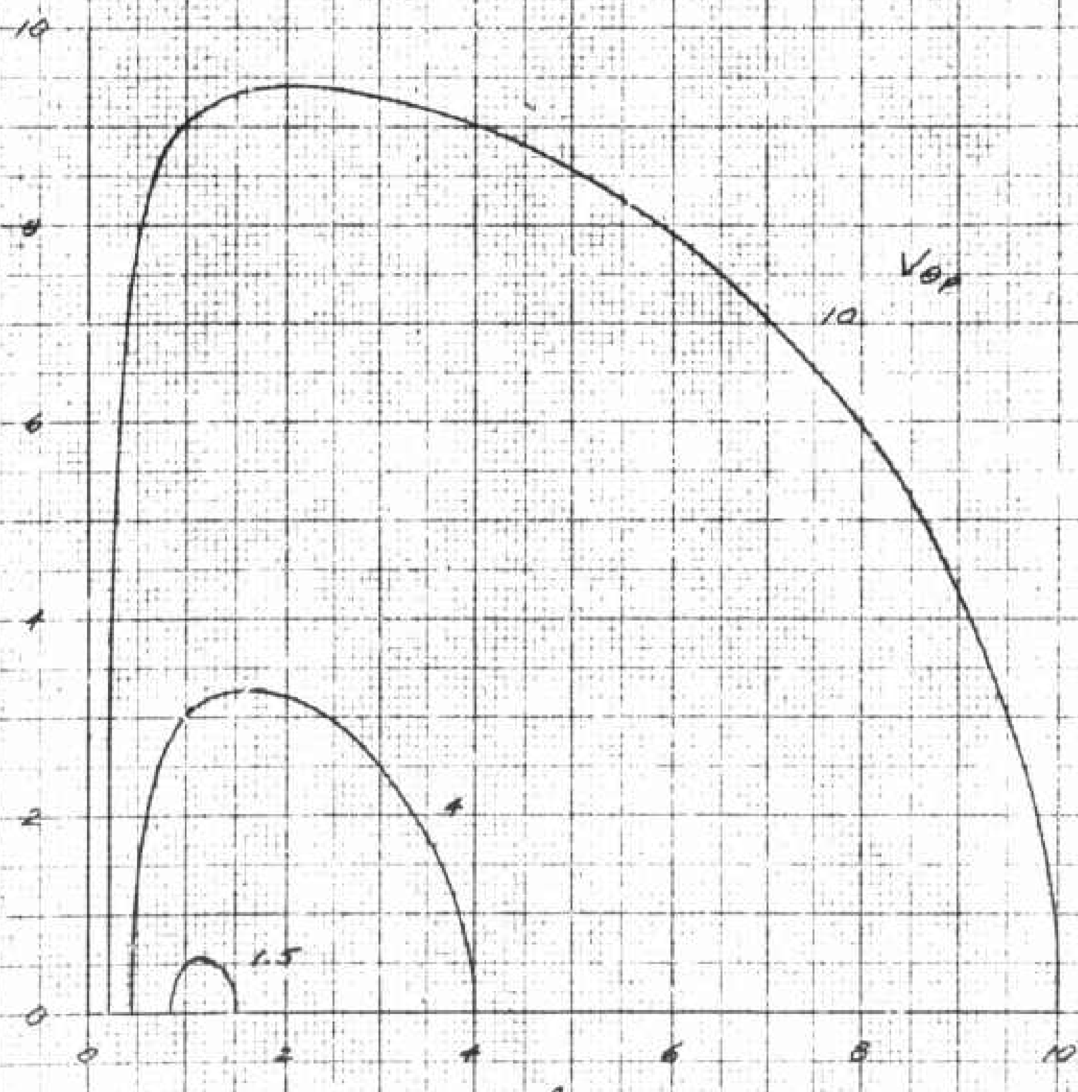


Figure 22. Perigee Velocity